All-Loop Four-Point Aharony-Bergman-Jafferis-Maldacena Amplitudes from Dimensional Reduction of the Amplituhedron

Song He,[1,2] Chia-Kai Kuo,[3,*] Zhenjie Li,[1,4] and Yao-Qi Zhang[1,4]

[1]*CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

[2]*School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study; International Centre for Theoretical Physics Asia-Pacific, Beijing/Hangzhou, China*

[3]*Department of Physics and Center for Theoretical Physics, National Taiwan University, Taipei 10617, Taiwan*

[4]*School of Physical Sciences, University of Chinese Academy of Sciences, No. 19A Yuquan Road, Beijing 100049, China*

(Received 28 April 2022; revised 30 September 2022; accepted 24 October 2022; published 23 November 2022)

We define a new geometry obtained from the all-loop amplituhedron in $\mathcal{N}=4$ SYM by reducing its four-dimensional external and loop momenta to three dimensions. Focusing on the simplest four-point case, we provide strong evidence that the canonical form of this "reduced amplituhedron" gives the all-loop integrand of the Aharony-Bergman-Jafferis-Maldacena four-point amplitude. In addition to various all-loop cuts manifested by the geometry, we present explicitly new results for the integrand up to five loops, which are much simpler than results in $\mathcal{N}=4$ SYM. One of the reasons for such all-loop simplifications is that only a very small fraction of the so-called negative geometries survives the dimensional reduction, which corresponds to bipartite graphs. Our results suggest an unexpected relation between four-point amplitudes in these two theories.

DOI: 10.1103/PhysRevLett.129.221604

Introduction.—The amplituhedron in planar $\mathcal{N}=4$ SYM [1–3] is arguably one of the most surprising mathematical structures of scattering amplitudes we have seen, where basic principles like locality and unitarity seem to have originated from the underlying geometric picture. All-loop integrands and tree amplitudes are given by the canonical form, which has logarithmic singularities only on boundaries of the amplituhedron [4,5]. As a beautiful mathematical object with remarkable physical properties, the amplituhedron has been extensively studied both at tree and loop level (cf. [2,6–14]), and in particular it can be used to make all-loop predictions about cuts of the integrand [15,16], which seem impossible otherwise. On the other hand, even for the four-point ($n=4$) L-loop amplituhedron, the geometry becomes more complicated as L increases, and an explicit computation for $L\geq 4$ becomes rather difficult (though $n=4$ integrand has been known to $L=10$ [17–20]). Moreover, despite various interesting ideas extending geometries beyond planar $\mathcal{N}=4$ SYM [21–31], an example of an all-loop amplituhedron in any other theory has yet to be found.

By dimensionally reducing external and loop (region) momenta of the amplituhedron, we obtain a reduced amplituhedron with rich structures, but the computation of canonical forms becomes greatly simplified, at least for the $n=4$ case, which is a $3L$-dimensional geometry in the space of L loop variables. Surprisingly, we find very strong evidence that this simplified $n=4$ geometry may be the long-sought-after all-loop amplituhedron for four-point amplitudes in $\mathcal{N}=6$ Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [32]. In spite of an extensive literature on ABJM amplitudes at tree and one-loop level (cf. [33–37]), much less is known about multiloop ABJM integrands beyond $L=2$ [38,39] (the only data available is a conjecture for $n=4$, $L=3$ in [40]); even at tree level, the amplituhedron in momentum space has only been proposed recently without obvious analog in momentum-twistor space yet [41,42]. In this Letter, we will not only show that the canonical forms of this $n=4$ reduced amplituhedron manifest various highly nontrivial all-loop cuts of ABJM amplitudes, but also push the frontier significantly by presenting compact expressions for ABJM integrands up to $L=5$.

In $\mathcal{N}=4$ SYM, it is beneficial to decompose the $n=4$ amplituhedron into building blocks called negative geometries [43], and at each loop, nontrivial negative geometries combine to give the integrand for an infrared-finite observable closely related to the logarithm of amplitudes (or equivalently Wilson loops with a single insertion) [44–49]. The analogous decomposition of the reduced amplituhedron reveals enormous simplifications from $D=4$ to $D=3$: only a tiny fraction of negative geometries, namely those corresponding to bipartite graphs, contribute to the integrand, with very simple pole structures. This lies at the heart of all-loop simplifications when reducing the $n = 4$ amplituhedron to $D = 3$.

Dimensional reduction of the amplituhedron.—In this section, we first give the definition of the reduced amplituhedron for $n = 4$, and then we outline a huge reduction of negative geometries in $D = 3$ of the corresponding geometries in $D = 4$ [43].

Definition of reduced amplituhedron:Recall that the n-point amplituhedron is defined in the space of n momentum twistors [50], Z_a^I with $a = 1, 2, \ldots, n$ for external kinematics, as well as L lines in the twistor space, $(AB)_i^{IJ}$ with $i = 1, \ldots, L$ for loop momenta; here $I, J = 1, \ldots, 4$ are $SL(4)$ indices, and the simplest bosonic $SL(4)$ invariant is defined as $\langle abcd\rangle \equiv \epsilon_{IJKL} Z_a^I Z_b^J Z_c^K Z_d^L$ (and similarly for $\langle (AB)_i ab\rangle$ and $\langle (AB)_i (AB)_j\rangle$). In [51], external kinematics in $D = 3$ was defined by dimensionally reducing every external line, $Z_a Z_{a+1}$; in a completely analogous manner, here we also need to dimensionally reduce all loop variables $(AB)_i$, both of which are achieved by the so-called symplectic conditions on these lines:

$$\mathbf{\Omega}_{IJ} Z_a^I Z_{a+1}^J = \mathbf{\Omega}_{IJ} A_i^I B_i^J = 0, \quad \text{with} \quad \mathbf{\Omega} = \begin{pmatrix} 0 & \epsilon_{2\times 2} \\ \epsilon_{2\times 2} & 0 \end{pmatrix} \tag{1}$$

for $a = 1, 2, \ldots, n$ and $i = 1, \ldots, L$, where the totally antisymmetric matrix is defined as $\epsilon_{2\times 2} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

We can define the reduced amplituhedron for any n and L by restricting the $D = 4$ amplituhedron geometry on the subspace given by (1), as long as it has a nonvanishing support there. In this Letter, we focus on the special case $n = 4$, and it is clear that we have a $3L$-dimensional geometry defined in the projected $(AB)_{i=1,\ldots,L}$ space. An important subtlety is that $\langle 1234\rangle < 0$ for real Z's satisfying symplectic conditions, thus we need to flip the overall sign for the definition of the $D = 4$ amplituhedron [1]: we require $\langle AB12\rangle, \langle AB23\rangle, \langle AB34\rangle, \langle AB14\rangle < 0$ and $\langle AB13\rangle, \langle AB24\rangle > 0$, for any loop (AB), as well as $\langle (AB)_i (AB)_j\rangle < 0$, all on the support of (1).

A convenient parametrization is $(AB)_i = (Z_1 + x_i Z_2 - w_i Z_4, y_i Z_2 + Z_3 + z_i Z_4)$ [2], and the symplectic condition on $(AB)_i$ becomes $x_i z_i + y_i w_i - 1 = 0$; the $n = 4$ geometry is defined by ($x_{i,j} \coloneqq x_i - x_j$, etc.)

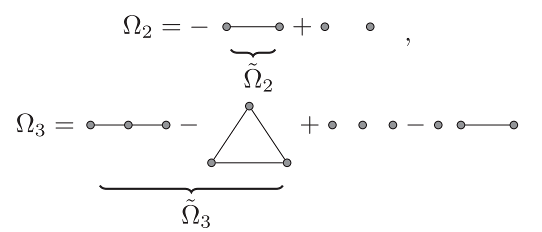
$$\begin{aligned} \forall\, i\colon\ x_i, y_i, z_i, w_i > 0, \qquad x_i z_i + y_i w_i = 1, \\ \forall\, i, j\colon\ x_{i,j} z_{i,j} + y_{i,j} w_{i,j} < 0. \end{aligned} \tag{2}$$

We denote this geometry as $\mathcal{A}_L$ with the canonical form $\Omega(\mathcal{A}_L) \coloneqq \Omega_L$, and our main claim is that Ω_L gives the L-loop planar integrand for four-point ABJM amplitudes (after stripping off the overall tree amplitude).

Negative geometries and their dimension reduction:In [43], a nice rewriting for the $n = 4$ amplituhedron [2] was proposed, where it is decomposed into a sum of negative geometries given by "mutual negativity" conditions, which trivially carries over to our $\mathcal{A}_L$ in $D = 3$; each negative geometry is represented by a labeled graph with L nodes and E edges (edge (ij) for $\langle (AB)_i (AB)_j\rangle > 0$ since we reversed all signs, and no condition otherwise), with an overall sign factor $(-)^E$. We sum over all graphs with L nodes without 2-cycles,

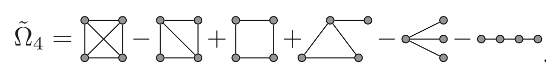
$$\mathcal{A}_L = \sum_g (-)^{E(g)} \mathcal{A}(g), \tag{3}$$

where $\mathcal{A}(g)$ is the (oriented) geometry for graph g. It suffices to consider all *connected* graphs, whose (signed) sum gives the geometry for the logarithm of amplitudes [43]. Such a decomposition is useful since each A_g is simpler, whose canonical form is easier to compute. The form for $L = 2$, 3 reads



(4)

where the connected part, or log of the amplitude, is denoted as $\tilde{\Omega}_L$, e.g., $\tilde{\Omega}_2 \coloneqq \Omega_2 - \frac{1}{2}\Omega_1^2$. Similarly, the connected part of $L = 4$ is given by the sum of graphs with six topologies (and so on for higher L),



(5)

What is new in $D = 3$ is that most of these geometries do not contribute at all: we find that remarkably, under dimensional reduction only those negative geometries with *bipartite* graphs survive in the decomposition. For example, for $\tilde{\Omega}_3$, the chain graph contributes but the triangle does not; while for $\tilde{\Omega}_4$, only the two kinds of tree graphs and the box contribute. This represents a major simplification as the fraction of bipartite graphs in all graphs tend to zero quickly as L increases: for $L = 2, \ldots, 7$, the number of topologies for connected graphs are 1,2,6,21,112,853, but that of bipartite topologies decrease to 1,1,3,5,17,44, e.g., for $\tilde{\Omega}_5$, only five topologies (out of 21) survive the reduction. Moreover, it turns out that one can compute the canonical form for geometries of bipartite graphs with relative ease, mainly due to their remarkably simple pole structures. The reduction and the computation of their forms will be explained in detail in [52].

ABJM integrands from reduced amplituhedron.—Let us take a first look at $L = 1$, where the geometry is defined as

$x, y, z, w > 0$ and $xz + yw = 1$. In this special case, its canonical form is nothing but reducing the $D = 4$ form, $(dx/x)(dy/y)(dz/z)(dw/w)$, onto the $D = 3$ subspace:

$$\Omega_1 = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w}\delta(xz + yw - 1), \tag{6}$$

and we can rewrite it in a covariant form

$$\Omega_1 = \frac{d^3AB\langle 1234\rangle^{3/2}(\langle AB13\rangle\langle AB24\rangle)^{1/2}}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}, \tag{7}$$

where the measure is

$$d^3AB \coloneqq \langle ABd^2A\rangle\langle ABd^2B\rangle\delta(\mathbf{\Omega}_{IJ}A^IB^J). \tag{8}$$

Here, the numerator, $(\langle AB13\rangle\langle AB24\rangle)^{1/2} \propto (xz + yw)^{1/2} = 1$, turns out to be proportional to the famous ϵ numerator of the dual conformal invariant box in $D = 3$ [38]. Thus, the dimensional reduction of the one-loop box in $\mathcal{N} = 4$ SYM gives the one-loop box with the ϵ numerator in ABJM, which confirms our claim at one loop.

All-loop soft and vanishing cuts.—Now we provide strong evidence that Ω_L gives and L-loop integrand for $L > 1$. Let us first rewrite the inequalities (2) by solving for variables $x_i = (1 - y_i w_i)/z_i$, and we arrive at the following equivalent definition

$$\begin{gathered} w_i, y_i, z_i > 0, w_i y_i < 1, \\ d_{i,j} \coloneqq (w_i z_j - w_j z_i)(y_i z_j - y_j z_i) - z_{i,j}^2 < 0, \end{gathered} \tag{9}$$

for $i, j = 1, \ldots, L$. Before proceeding to explicit computations, we see that (9) allows us to study some all-loop cuts in a simple way. An important cut of four-point ABJM amplitudes is the so-called soft cut, where we take $\langle AB12\rangle = \langle AB23\rangle = \langle AB34\rangle = 0$, or equivalently $y = z = w = 0$ for any given loop, and the result is the $(L-1)$-loop integrand. From geometry, with $y_i = z_i = w_i = 0$ clearly $d_{i,j} < 0$ is trivially satisfied for any $j \neq i$, thus the geometry reduces to the $(L-1)$-loop one: $\partial^{(3)}_{y_i=z_i=w_i=0}\mathcal{A}_L = \mathcal{A}_{L-1}$. The soft cut is satisfied.

Moreover, certain cuts are known to vanish due to the presence of vanishing odd-point amplitudes: by cutting $\langle (AB)_i 12\rangle = \langle (AB)_i (AB)_j\rangle = \langle (AB)_j 12\rangle = 0$ (or changing the last one to $\langle (AB)_j 34\rangle$), we isolate a three-point (or five-point, respectively) amplitude, which must vanish. These are equivalent to setting $w_i = d_{i,j} = w_j = 0$ or $w_i = d_{i,j} = y_j = 0$; in either case, $d_{i,j} = -z_{i,j}^2 < 0$ is trivially satisfied on the support of the other two conditions, and the residue vanishes as expected. These and other vanishing cuts are nicely guaranteed by the geometry. We have checked our new results for $L = 4$, 5 thoroughly: in addition to various all-loop checks, we have computed unitarity cuts and checked that Ω_4 and Ω_5 satisfy the optical theorem: double cuts are given by products of various lower-loop integrands.

Explicit results up to five loops:We present explicitly the canonical form up to five loops and leave the detailed derivation in Supplemental Material [52]. To save space, we introduce a shorthand notation $(AB)_i \coloneqq \ell_i$, and it turns out that the logarithm $\tilde{\Omega}_2$ is simply

$$\tilde{\Omega}_2 = -2\frac{d^3\ell_1 d^3\ell_2\langle 1234\rangle^2}{\langle \ell_1 12\rangle\langle \ell_1 34\rangle\langle \ell_1\ell_2\rangle\langle \ell_2 23\rangle\langle \ell_2 14\rangle} + (\ell_1 \leftrightarrow \ell_2). \tag{10}$$

This is nothing but a double-triangle integrand where external region momenta correspond to (12),(34) for ℓ_1, and (23),(14) for ℓ_2, and vice versa. One can easily check that by adding back one-loop squared, we recover the well-known two-loop result [38,39].

One interesting feature of $\tilde{\Omega}_2$ is that for each term, ℓ_1 contains only two poles, $\langle \ell_1 12\rangle\langle \ell_1 34\rangle$ or $\langle \ell_1 23\rangle\langle \ell_1 14\rangle$ (similarly for ℓ_2). We denote these combinations and mutual conditions, which include all possible poles, as

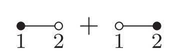
$$s_i \coloneqq \langle \ell_i 12\rangle\langle \ell_i 34\rangle, \quad t_i \coloneqq \langle \ell_i 23\rangle\langle \ell_i 14\rangle, \quad D_{i,j} \coloneqq -\langle \ell_i \ell_j\rangle.$$

We denote the ϵ numerator and ℓ-independent factor as

$$\epsilon_i \coloneqq (\langle \ell_i 13\rangle\langle \ell_i 24\rangle\langle 1234\rangle)^{1/2}, \qquad c \coloneqq \langle 1234\rangle,$$

and also the integrand with measure $\prod_{i=1}^L d^3\ell_i$ stripped off as $\underline{\tilde{\Omega}}_L$. For $L = 1$, 2, they read

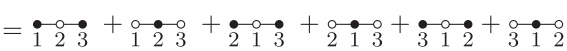
$$\begin{aligned} \underline{\Omega}_1 &= \frac{c\epsilon_1}{s_1 t_1}, \\ \underline{\tilde{\Omega}}_2 &= \frac{2c^2}{D_{1,2}}\left(\frac{1}{s_1 t_2} + \frac{1}{t_1 s_2}\right) = \end{aligned} \tag{11}$$



where the $L = 2$ case is represented by "chain" graphs with s, t pole structures represented by black and white coloring, respectively.

Now we are ready to move to $L = 3$. Remarkably we find that $\tilde{\Omega}_3$ only receives contributions with pole structures of three chains (each with two possible choices of s and t), as represented by six bipartite graphs. It reads

$$\underline{\tilde{\Omega}}_3 =$$



$$= \frac{4c^2\epsilon_2}{s_1 t_2 s_3 D_{1,2} D_{2,3}} + (s \leftrightarrow t) + 2\text{ perms.}, \tag{12}$$

where for each chain we have two terms with s, t swapped, and similar to $L = 1$ the ϵ numerator makes correct weight: $\underline{\tilde{\Omega}}_L$ has degree (-3) in each ℓ_i. Each term is again a ladder integral with two triangles and a middle box (with ϵ numerator). Very nontrivially, when converting back to Ω_3, it agrees with the conjecture from generalized unitarity

[40]; various independent checks are presented in Supplemental Material [52].

After having familiarized ourselves with the notation, we present the $L=4$ result in a very compact form, and it turns out $\tilde{\Omega}_4$ only gets contributions from three topologies. We give all bipartite graphs in Fig. 1. The first type consists of 12×2 bipartite chain graphs

$$C = 8c^2\frac{\epsilon_2\epsilon_3}{D_{1,2}D_{2,3}D_{3,4}s_1t_2s_3t_4} + (s\leftrightarrow t) + 11\text{ perms}; \tag{13}$$

then we have 4×2 "star" bipartite graphs

$$S = 8c^3\frac{t_1}{D_{1,2}D_{1,3}D_{1,4}s_1t_2t_3t_4} + (s\leftrightarrow t) + 3\text{ perms}; \tag{14}$$

finally, we have three box bipartite graphs:

$$B = 4\frac{4\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - c(\epsilon_1\epsilon_3N^t_{24} + \epsilon_2\epsilon_4N^s_{13}) - c^2N^{\text{cyc}}_{1,2,3,4}}{D_{1,2}D_{2,3}D_{3,4}D_{4,1}s_1t_2s_3t_4} + (s\leftrightarrow t) + 2\text{ perms}, \tag{15}$$

where we define combinations similar to s, t for two ℓ s,

$$N^s_{13} := \langle\ell_1 12\rangle\langle\ell_3 34\rangle + \langle\ell_3 12\rangle\langle\ell_1 34\rangle,$$
$$N^t_{24} := \langle\ell_2 14\rangle\langle\ell_4 23\rangle + \langle\ell_4 14\rangle\langle\ell_2 23\rangle,$$
$$N^{\text{cyc}}_{i,j,k,l} := \langle\ell_i 12\rangle\langle\ell_j 34\rangle\langle\ell_k 12\rangle\langle\ell_l 34\rangle + \text{cyc}(1,2,3,4), \tag{16}$$

where cyc(1,2,3,4) indicates cyclic rotations of dual points $12\to 23\to 34\to 14$; $(s\leftrightarrow t)$ denotes the symmetrization in the pairs (12,34) and (23,14).

The final result for $L=4$ reads

$$\underline{\tilde{\Omega}}_4 = -C - S + B, \tag{17}$$

where the signs are given by $(-)^E$ with E the number of edges. We compute these forms using a method whose details are given in Supplemental Material [52]: after writing down denominators according to bipartite graphs, we use an ansatz for each numerator which consists of all possible terms consistent with symmetries, and fix all parameters (with numerous cross checks) from various boundaries whose canonical forms can be computed directly.

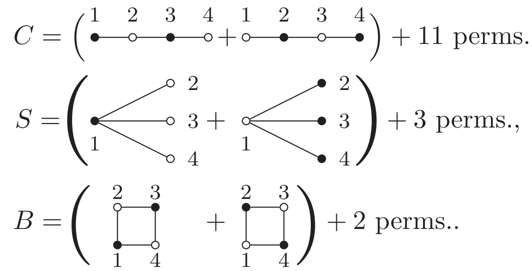
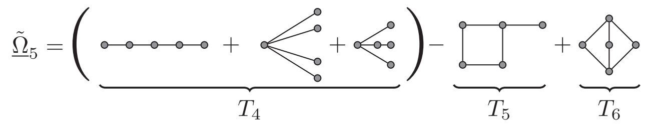


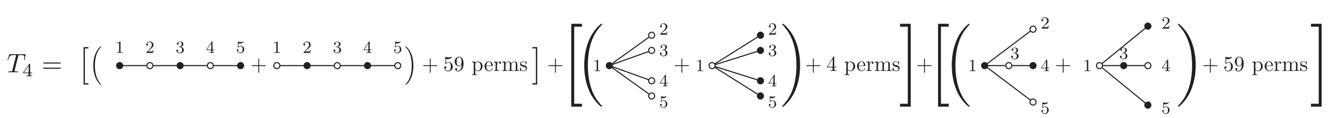
FIG. 1. All bipartite graphs contributing to $\tilde{\Omega}_4$.

Finally, we compute the five-loop form $\tilde{\Omega}_5$, which consists of five topologies: three tree graphs with four edges, a box with an external line (five edges), and one with two nodes connected to three nodes (six edges). We have



(18)

where only graphs without labels or color are shown, and T_m denotes the total contribution from graphs with m edges. The contribution of all trees, T_4, takes similar forms as lower trees (e.g., C, S for $L=4$). We follow the same method for computing the remaining contributions: 60×2 bipartite graphs for T_5 and 10×2 for T_6, which are analogous to B (especially T_5 takes a very similar form). The upshot is that the full five-loop result boils down to just these two new functions (T_5 and T_6) given as follows:



$$= \left(\frac{16c^3\epsilon_2\epsilon_4}{s_1t_2s_3t_4s_5D_{1,2}D_{2,3}D_{3,4}D_{4,5}} + (s\leftrightarrow t) + 59\,\text{perms}\right) + \left(\frac{16c^3\epsilon_1 s_1}{t_1s_2s_3s_4s_5D_{1,2}D_{1,3}D_{1,4}D_{1,5}} + (s\leftrightarrow t) + 4\,\text{perms}\right)$$
$$+ \left(\frac{16c^3\epsilon_3t_1}{s_1t_2t_3s_4t_5D_{1,2}D_{1,3}D_{3,4}D_{1,5}} + (s\leftrightarrow t) + 59\,\text{perms}\right),$$

$$T_5 = 8c\frac{4\epsilon_1\epsilon_3\epsilon_4s_2 - \epsilon_1\epsilon_2\epsilon_3N^t_{24} - c(-\epsilon_1t_2N^t_{34} - \epsilon_3t_2N^t_{14} + \epsilon_4s_2N^s_{13} + \epsilon_2N^{\text{cyc}}_{1,2,3,4})}{s_1t_2s_3t_4s_5D_{1,2}D_{2,3}D_{3,4}D_{4,1}D_{2,5}} + (s\leftrightarrow t) + 59\text{ perms},$$

$$T_6 = \frac{4}{cs_1t_2t_3t_4s_5D_{1,2}D_{1,3}D_{1,4}D_{2,5}D_{3,5}D_{4,5}}\big(-8\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5N^s_{15} + c\epsilon_2\epsilon_3\epsilon_4P_a$$
$$+ c[\epsilon_1\epsilon_2\epsilon_3P_b + (\ell_1\leftrightarrow\ell_5) + \epsilon_1\epsilon_2\epsilon_5P_c + \text{cyc}(\ell_2,\ell_3,\ell_4)]$$
$$+ c^2[\epsilon_1P_d + (\ell_1\leftrightarrow\ell_5)] + c^2[\epsilon_2P_e + \text{cyc}(\ell_2,\ell_3,\ell_4)]\big) + (s\leftrightarrow t) + 9\text{ perms}; \tag{19}$$

where in T_6, we have polynomials $P_a, P_b, \ldots, P_e$ with certain weights in $\ell_1, \ldots, \ell_5$, and here we record their expressions:

$$P_a := -20 s_1 s_5 + 16 t_1 t_5 + (N^s_{15})^2, \qquad P_b := 6 s_5 N^s_{14}, \qquad P_c := N^s_{15} N^t_{34} - 4 N^{\rm cyc}_{1,3,5,4}, \tag{20}$$

$$\begin{aligned} P_d := & -s_5[N^s_{12}N^t_{34} + \mathrm{cyc}(\ell_2,\ell_3,\ell_4)] + [2\langle \ell_5 12\rangle^2 \langle \ell_1 12\rangle\langle \ell_2 34\rangle\langle \ell_3 34\rangle\langle \ell_4 34\rangle + \mathrm{cyc}(1,2,3,4)] \\ & + 2t_5\{\langle \ell_1 14\rangle[\langle \ell_2 14\rangle\langle \ell_3 23\rangle\langle \ell_4 23\rangle + \mathrm{cyc}(\ell_2,\ell_3,\ell_4)] + (14 \leftrightarrow 23)\}, \end{aligned} \tag{21}$$

$$\begin{aligned} P_e := & \, 2 s_1 s_5 (N^t_{34} - N^s_{34}) - 4 t_1 t_5 N^t_{34} - \{s_5[\langle \ell_1 12\rangle^2 \langle \ell_3 34\rangle\langle \ell_4 34\rangle + (12 \leftrightarrow 34)] + (\ell_1 \leftrightarrow \ell_5)\} \\ & + N^s_{15}[\langle \ell_1 14\rangle\langle \ell_5 14\rangle\langle \ell_3 23\rangle\langle \ell_4 23\rangle + (14 \leftrightarrow 23)]. \end{aligned} \tag{22}$$

Conclusions and outlook.—In this Letter, we have discovered a surprising connection between four-point amplitudes in $\mathcal{N}=4$ SYM and ABJM: by dimensionally reducing from $D=4$ to $D=3$, the amplituhedron of the former becomes that of the latter, which we have checked explicitly to five loops and for various all-loop cuts. The reduced geometries exhibit remarkable structures and simplicity.

One pressing question is the physical meaning of reduced amplituhedra for higher points: does it correspond to ABJM amplitudes or certain null polygonal Wilson loops [39,54]? On the other hand, our $n=4$ integrand clearly contains higher-point ones via unitarity, e.g., their single cuts give a forward limit of six-point amplitudes [55,56] (see Ref. [57] for $L=2$). It would be fascinating to compute such higher-point forms at $L \geq 3$ and reveal possible geometries [58].

Last but not least, integrating the forms produces an interesting, finite observable in ABJM theory (analogous to that in $\mathcal{N}=4$ SYM [43]). It is straightforward to do so for $L \leq 3$, and we expect to extract that cusp anomalous dimension of ABJM from it; we also expect that resummation for (some of) bipartite geometries would allow us to study their contributions nonperturbatively. We will report these results elsewhere [59].

We thank Nima Arkani-Hamed, Yu-tin Huang, Qinglin Yang for inspiring discussions, and especially Yu-tin Huang for clarifying the ϵ structure of ABJM. The research of S. H. is supported in part by the National Natural Science Foundation of China under Grants No. 11935013, No. 11947301, No. 12047502, No. 12047503. The research of C.-K. K. is supported by Taiwan Ministry of Science and Technology Grant No. 109-2112-M-002-020-MY3.

*Corresponding author.
chiakaikuo@gmail.com

[1] N. Arkani-Hamed and J. Trnka, The amplituhedron, J. High Energy Phys. 10 (2014) 030.
[2] N. Arkani-Hamed and J. Trnka, Into the amplituhedron, J. High Energy Phys. 12 (2014) 182.
[3] N. Arkani-Hamed, H. Thomas, and J. Trnka, Unwinding the amplituhedron in binary, J. High Energy Phys. 01 (2018) 016.
[4] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes* (Cambridge University Press, Cambridge, England, 2016), 10.1017/CBO9781316091548.
[5] N. Arkani-Hamed, Y. Bai, and T. Lam, Positive geometries and canonical forms, J. High Energy Phys. 11 (2017) 039.
[6] T. Lam, Amplituhedron cells and stanley symmetric functions, Commun. Math. Phys. **343**, 1025 (2016).
[7] S. N. Karp and L. K. Williams, The $m=1$ amplituhedron and cyclic hyperplane arrangements, Int. Math. Res. Not. **2019**, 1401 (2019).
[8] P. Galashin and T. Lam, Parity duality for the amplituhedron, Compos. Math. **156**, 2207 (2020).
[9] S. Franco, D. Galloni, A. Mariotti, and J. Trnka, Anatomy of the amplituhedron, J. High Energy Phys. 03 (2015) 128.
[10] D. Galloni, Positivity Sectors and the Amplituhedron, arXiv:1601.02639.
[11] Y. Bai and S. He, The amplituhedron from momentum twistor diagrams, J. High Energy Phys. 02 (2015) 065.
[12] Y. Bai, S. He, and T. Lam, The amplituhedron and the one-loop Grassmannian measure, J. High Energy Phys. 01 (2016) 112.
[13] J. Rao, 4-particle amplituhedronics for 3-5 loops, Nucl. Phys. **B943**, 114625 (2019).
[14] L. Ferro, T. Łukowski, and M. Parisi, Amplituhedron meets Jeffrey–Kirwan residue, J. Phys. A **52**, 045201 (2019).
[15] N. Arkani-Hamed, C. Langer, A. Y. Srikant, and J. Trnka, Deep Into the Amplituhedron: Amplitude Singularities at All Loops and Legs, Phys. Rev. Lett. **122**, 051601 (2019).
[16] C. Langer and A. Yelleshpur Srikant, All-loop cuts from the amplituhedron, J. High Energy Phys. 04 (2019) 105.
[17] Z. Bern, J. J. M. Carrasco, H. Johansson, and D. A. Kosower, Maximally supersymmetric planar Yang-Mills amplitudes at five loops, Phys. Rev. D **76**, 125020 (2007).
[18] J. L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin, and A. Volovich, The soft-collinear bootstrap: $\mathcal{N}=4$ Yang-Mills amplitudes at six- and seven-loops, J. High Energy Phys. 03 (2012) 032.
[19] J. L. Bourjaily, P. Heslop, and V.-V. Tran, Perturbation Theory at Eight Loops: Novel Structures and the Breakdown of Manifest Conformality in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **116**, 191602 (2016).

[20] J. L. Bourjaily, P. Heslop, and V.-V. Tran, Amplitudes and correlators to ten loops using simple, graphical bootstraps, J. High Energy Phys. 11 (2016) 125.
[21] Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, and J. Trnka, Evidence for a nonplanar amplituhedron, J. High Energy Phys. 06 (2016) 098.
[22] N. Arkani-Hamed, P. Benincasa, and A. Postnikov, Cosmological Polytopes and the Wavefunction of the Universe, arXiv:1709.02813.
[23] N. Arkani-Hamed and P. Benincasa, On the Emergence of Lorentz Invariance and Unitarity from the Scattering Facet of Cosmological Polytopes, arXiv:1811.01125.
[24] N. Arkani-Hamed, Y. Bai, S. He, and G. Yan, Scattering forms and the positive geometry of kinematics, color and the worldsheet, J. High Energy Phys. 05 (2018) 096.
[25] G. Salvatori, 1-loop amplitudes from the Halohedron, J. High Energy Phys. 12 (2019) 074.
[26] N. Arkani-Hamed, S. He, G. Salvatori, and H. Thomas, Causal Diamonds, Cluster Polytopes and Scattering Amplitudes, arXiv:1912.12948.
[27] N. Arkani-Hamed, S. He, and T. Lam, Stringy canonical forms, J. High Energy Phys. 02 (2021) 069.
[28] N. Arkani-Hamed, S. He, T. Lam, and H. Thomas, Binary Geometries, Generalized Particles and Strings, and Cluster Algebras, arXiv:1912.11764.
[29] S. He and C. Zhang, Notes on scattering amplitudes as differential forms, J. High Energy Phys. 10 (2018) 054.
[30] D. Damgaard, L. Ferro, T. Lukowski, and M. Parisi, The momentum amplituhedron, J. High Energy Phys. 08 (2019) 042.
[31] D. Damgaard, L. Ferro, T. Lukowski, and R. Moerman, Momentum amplituhedron meets kinematic associahedron, J. High Energy Phys. 02 (2021) 041.
[32] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, $\mathcal{N}=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, J. High Energy Phys. 10 (2008) 091.
[33] Y.-t. Huang and A. E. Lipstein, Dual superconformal symmetry of $\mathcal{N}=6$ Chern-Simons theory, J. High Energy Phys. 11 (2010) 076.
[34] D. Gang, Y.-t. Huang, E. Koh, S. Lee, and A. E. Lipstein, Tree-level recursion relation and dual superconformal symmetry of the ABJM theory, J. High Energy Phys. 03 (2011) 116.
[35] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, One loop amplitudes in ABJM, J. High Energy Phys. 07 (2012) 029.
[36] A. Brandhuber, G. Travaglini, and C. Wen, All one-loop amplitudes in $\mathcal{N}=6$ superconformal Chern-Simons theory, J. High Energy Phys. 10 (2012) 145.
[37] T. Bargheer, N. Beisert, F. Loebbert, and T. McLoughlin, Conformal anomaly for amplitudes in $\mathcal{N}=6$ superconformal Chern–Simons theory, J. Phys. A **45**, 475402 (2012).
[38] W.-M. Chen and Y.-t. Huang, Dualities for loop amplitudes of $\mathcal{N}=6$ Chern-Simons matter theory, J. High Energy Phys. 11 (2011) 057.
[39] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, Scattering amplitudes/Wilson loop duality in ABJM theory, J. High Energy Phys. 01 (2012) 056.
[40] M. S. Bianchi and M. Leoni, On the ABJM four-point amplitude at three loops and BDS exponentiation, J. High Energy Phys. 11 (2014) 077.
[41] Y.-t. Huang, R. Kojima, C. Wen, and S.-Q. Zhang, The orthogonal momentum amplituhedron and ABJM amplitudes, J. High Energy Phys. 01 (2022) 141.
[42] S. He, C.-K. Kuo, and Y.-Q. Zhang, The momentum amplituhedron of SYM and ABJM from twistor-string maps, J. High Energy Phys. 02 (2022) 148.
[43] N. Arkani-Hamed, J. Henn, and J. Trnka, Nonperturbative negative geometries: amplitudes at strong coupling and the amplituhedron, J. High Energy Phys. 03 (2022) 108.
[44] L. F. Alday, E. I. Buchbinder, and A. A. Tseytlin, Correlation function of null polygonal Wilson loops with local operators, J. High Energy Phys. 09 (2011) 034.
[45] O. T. Engelund and R. Roiban, On correlation functions of Wilson loops, local and non-local operators, J. High Energy Phys. 05 (2012) 158.
[46] L. F. Alday, J. M. Henn, and J. Sikorowski, Higher loop mixed correlators in $\mathcal{N}=4$ SYM, J. High Energy Phys. 03 (2013) 058.
[47] J. M. Henn, G. P. Korchemsky, and B. Mistlberger, The full four-loop cusp anomalous dimension in $\mathcal{N}=4$ super Yang-Mills and QCD, J. High Energy Phys. 04 (2020) 018.
[48] D. Chicherin and J. M. Henn, Symmetry properties of Wilson loops with a Lagrangian insertion, J. High Energy Phys. 07 (2022) 057.
[49] D. Chicherin and J. Henn, Pentagon Wilson loop with Lagrangian insertion at two loops in $\mathcal{N}=4$ super Yang-Mills theory, J. High Energy Phys. 07 (2022) 038.
[50] A. Hodges, Eliminating spurious poles from gauge-theoretic amplitudes, J. High Energy Phys. 05 (2013) 135.
[51] H. Elvang, Y.-t. Huang, C. Keeler, T. Lam, T. M. Olson, S. B. Roland, and D. E. Speyer, Grassmannians for scattering amplitudes in 4d $\mathcal{N}=4$ SYM and 3d ABJM, J. High Energy Phys. 12 (2014) 181.
[52] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.221604 for the detail of the reduction and the computation of their forms, which includes Ref. [53].
[53] R. P. Stanley, Acyclic orientations of graphs, Discrete Math. **5**, 171 (1973).
[54] J. M. Henn, J. Plefka, and K. Wiegandt, Light-like polygonal Wilson loops in 3d Chern-Simons and ABJM theory, J. High Energy Phys. 08 (2010) 032; 11 (2011) 53.
[55] Y.-T. Huang and C. Wen, ABJM amplitudes and the positive orthogonal Grassmannian, J. High Energy Phys. 02 (2014) 104.
[56] Y.-t. Huang, C. Wen, and D. Xie, The positive orthogonal Grassmannian and loop amplitudes of ABJM, J. Phys. A **47**, 474008 (2014).
[57] S. Caron-Huot and Y.-t. Huang, The two-loop six-point amplitude in ABJM theory, J. High Energy Phys. 03 (2013) 075.
[58] S. He, Y.-t. Huang, and C.-K. Kuo (to be published).
[59] S. He, C.-K. Kuo, Z. Li, and Y.-Q. Zhang (to be published).