# Discrete Anomaly in 2D Conformal Field Theories

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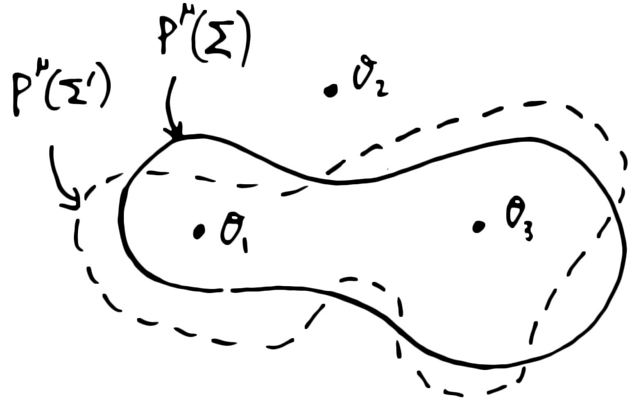
# Global Symmetry

- Global symmetries are powerful in constraining correlation functions and other physical observables in quantum field theory (QFT).
  - Ward identity, representations, selection rules…

- \`t Hooft anomaly is an obstruction for gauging a global symmetry. It constrains the dynamics and phases of QFT.
  [‘t Hooft anomaly is not bad but physical->ABJ]
  - ’t Hooft anomaly matching, RG flows, boundaries and interfaces…

- Symmetry transformations on operators can be implemented by topological operators →$U_g(M)$ topological defects

# Topological Defects




- Ward identity

$$\partial_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = -\sum_i \delta(x-x_i)\partial_i^\nu \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle$$

$$P^\nu(\Sigma) \equiv -\int_\Sigma dS_\mu T^{\mu\nu}(x)$$

- Continuous U(1) symmetry → $U_g(M) = e^{i\theta Q}$
  - topological property follows from the current conservation
- In general (continuous or discrete), a (0-form) global symmetry $g \in G$ is associated to a codimension-1 topological defect $L_g$.
- Symmetry transformation on local operators can be implemented by topological defect surrounding. In 2d, $L_g$ is a line.

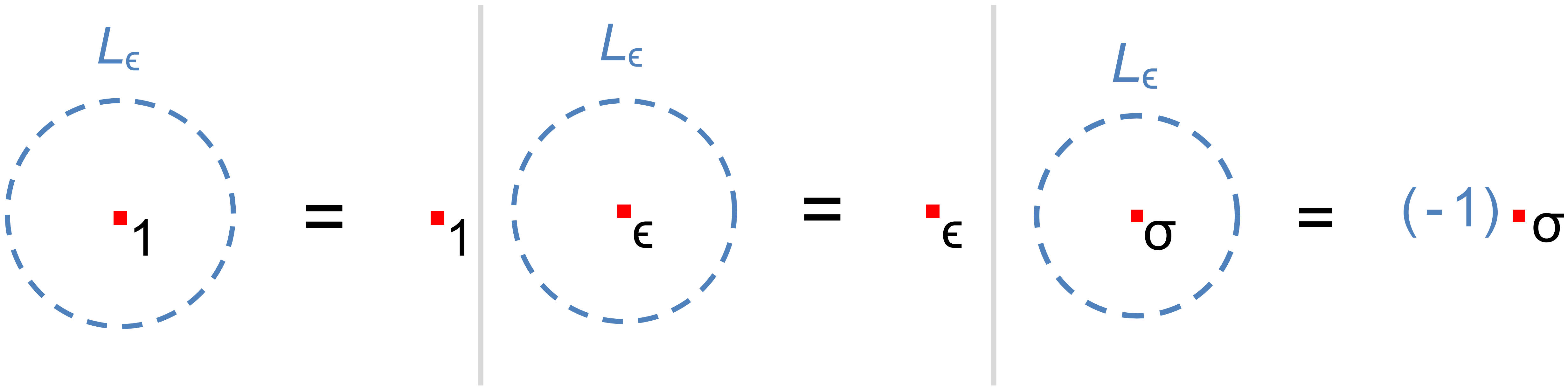
# 2D Ising Model (c=1/2)

- **3** primary operators in the Ising model and there is a $Z_2$ global symmetry.

Primary: $\mathbf{1}_{(0,0)}$ $\boldsymbol{\varepsilon}_{\left(\frac{1}{2},\frac{1}{2}\right)}$ $\boldsymbol{\sigma}_{\left(\frac{1}{16},\frac{1}{16}\right)}$

$\downarrow$

$Z_2$ Line: $\boldsymbol{L}_{\boldsymbol{\varepsilon}}$

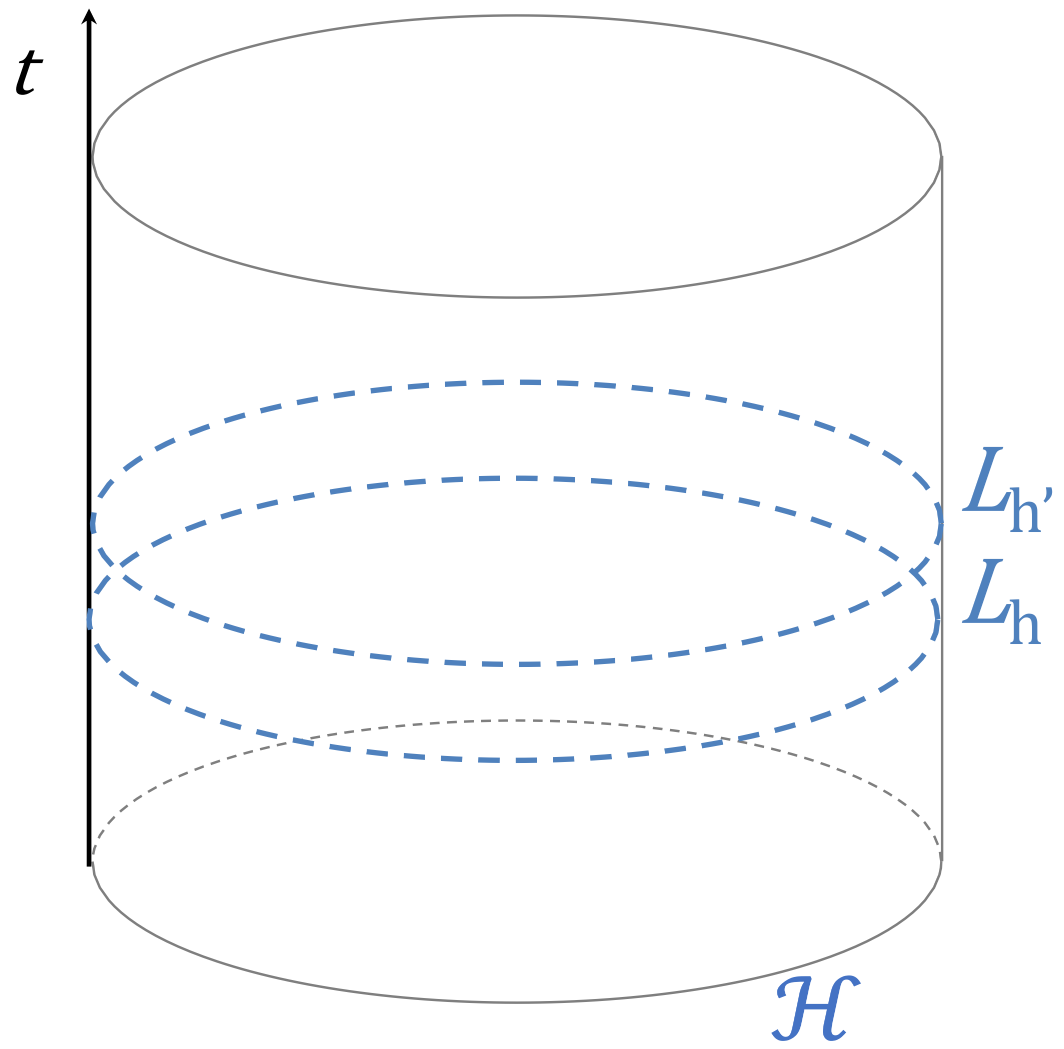



- $\mathrm{Z}_2$ symmetry transformation is implemented by $\boldsymbol{L_\varepsilon}$ .

# Symmetry Lines and Partition Function

- Symmetry lines act as symmetry transformation of local operator.
- The fusion of symmetry lines follows the group multiplication rules.
- The (inserted) torus partition function is

$$Z_{(h,1)}(\tau) = \mathrm{Tr}_{\mathcal{H}}[\hat{h} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}]$$



# Symmetry Lines and Partition Function

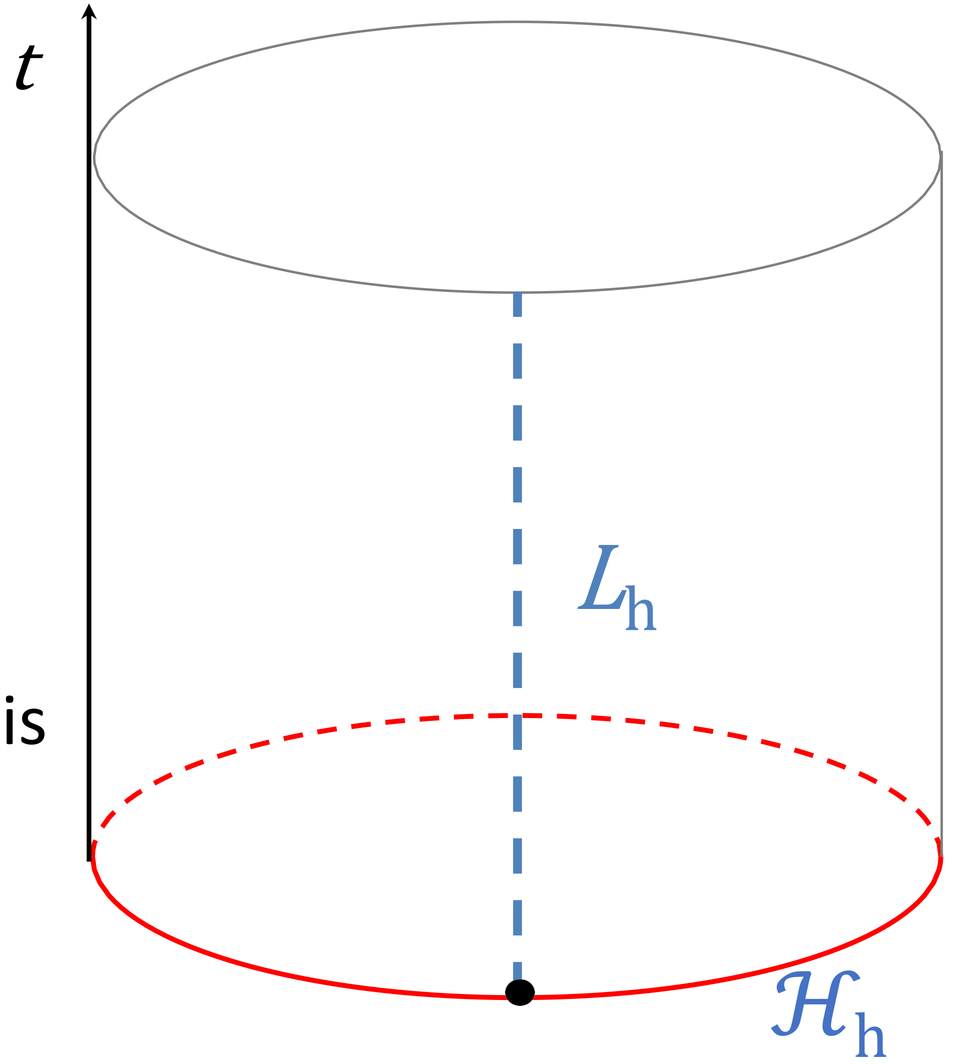
- Symmetry lines twist the boundary condition in quantization, therefore modified the Hilbert space.

- This twisting is related to automorphism of operator algebra

- The (twisted') torus partition function is

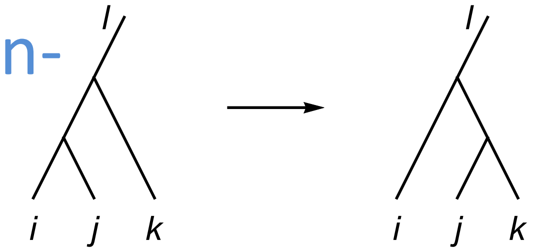
$$Z_{(1,h)}(\tau) = \mathrm{Tr}_{\mathcal{H}_h}[q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}]$$

$$SZ_{(h,1)}(\tau) = Z_{(1,h)}(\tau)$$

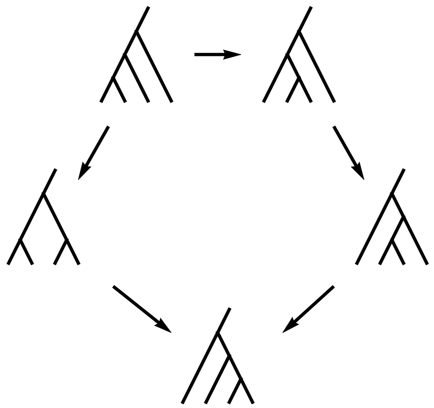


# How to detect ‘t Hooft anomaly for a global symmetry G in 2d CFT?

- Solving crossing phases (F-symbol) of TDLs [Chang-Lin-Shao-Yin, Bhardwaj-Tachikawa]




- F-symbol anomaly of G are classified/constrained by the pentagon identity → group cohomology $H^3(G,U(1))$ [Dijkgraaf-Witten,Chen-Gu-Liu-Wen]



- Our main result is that: [K.Kikuchi-YZ]

A new mixed anomaly=noncommutativity of symmetry line insertions on torus

# Consistency Conditions I

- Untwisted partition function is modular S and T invariant.

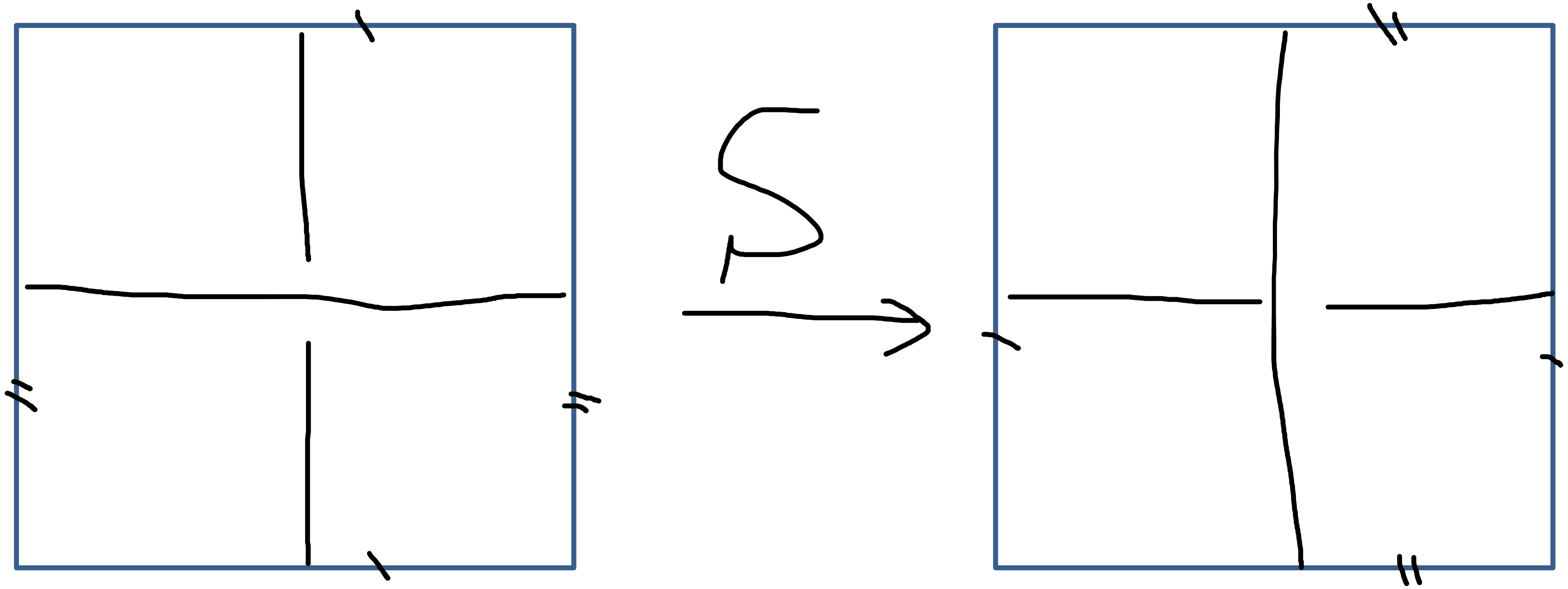
- If **2** circles on torus inserted by the same line, it is expected to be modular S-invariant

$$SZ_{(h,h)}(\tau) = Z_{(h,h)}(\tau)$$

- In general, when **2** circles inserted by two different lines, we should have

$$SZ_{(h,h')} = Z_{(h',h)}$$

$\mathbb{Z}_2$



# Ising Model (c=1/2)

$$Z_{(h,1)}(\tau) = |\chi_{id}(\tau)|^2 + |\chi_{\varepsilon}(\tau)|^2 - |\chi_{\sigma}(\tau)|^2$$

$$Z_{(1,h)}(\tau) = \bar{\chi}_{id}(\bar{\tau})\chi_{\varepsilon}(\tau) + \bar{\chi}_{\varepsilon}(\bar{\tau})\chi_{id}(\tau) + |\chi_{\sigma}(\tau)|^2$$

$$Z_{(h,h)}(\tau) = -\bar{\chi}_{id}(\bar{\tau})\chi_{\varepsilon}(\tau) - \bar{\chi}_{\varepsilon}(\bar{\tau})\chi_{id}(\tau) + \bar{\chi}_{\sigma}(\bar{\tau})\chi_{\sigma}(\tau)$$

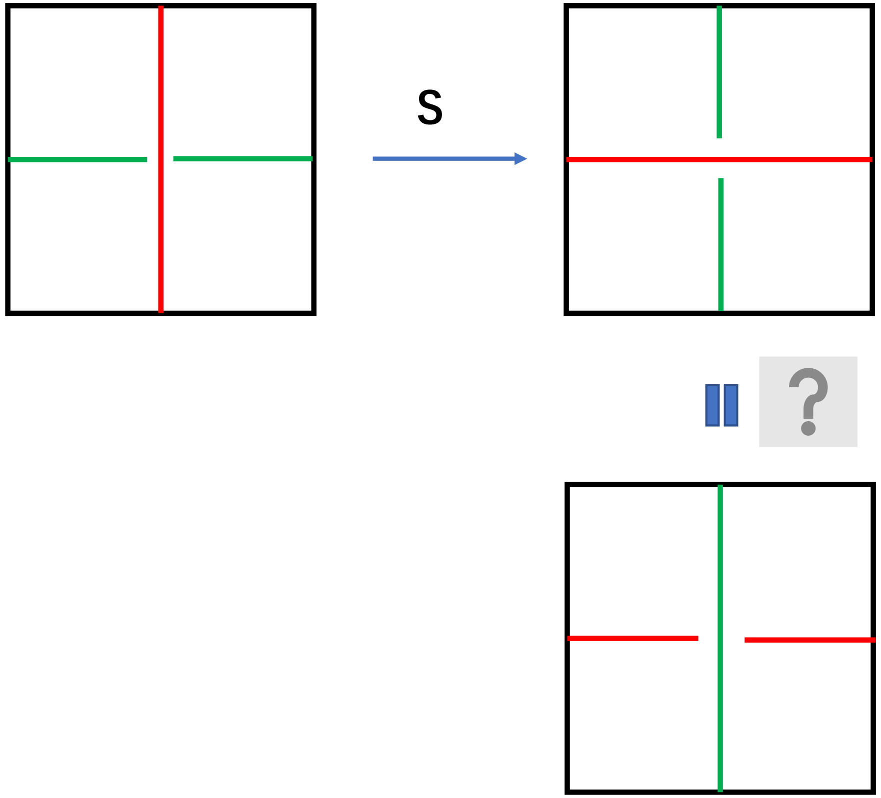
$$SZ_{(h,h)}(\tau) = Z_{(h,h)}(\tau)$$ [Lin-Shao, Kikuchi-YZ]

# SU(2)_k Wess-Zumino-Witten (WZW)

$$Z_{(\mathbf{1},h)} = \mathcal{S} Z_{(h,\mathbf{1})} = \sum_{\hat{\mu} \in P_+^k} \bar{\chi}_{A\hat{\mu}} \chi_{\hat{\mu}}$$

$$\begin{aligned}
SZ_{(h,h)}(\tau) &= S \sum_{j=0,1/2,\ldots,k/2} (-i)^k (-)^{2j} \chi_j(\tau) \bar{\chi}_{\frac{k}{2}-j}(\bar{\tau}) \\
&= \sum_{j,j_1,j_2} (-i)^k (-)^{2j} S_{jj_1} S_{\frac{k}{2}-j,j_2} \chi_{j_1}(\tau) \bar{\chi}_{j_2}(\bar{\tau}) \\
&= \sum_{j,j_1,j_2} (-i)^k (-)^{2j} S_{jj_1} (-)^{2j_2} S_{jj_2} \chi_{j_1}(\tau) \bar{\chi}_{j_2}(\tau) \\
&= \sum_{j,j_1,j_2} (-i)^k (-)^{2j_2} S_{jj_1} S_{j,\frac{k}{2}-j_2} \chi_{j_1}(\tau) \bar{\chi}_{j_2}(\bar{\tau}) \\
&= \sum_{j_1,j_2} (-i)^k (-)^{2j_2} \delta_{j_1,\frac{k}{2}-j_2} \chi_{j_1}(\tau) \bar{\chi}_{j_2}(\tau) \\
&= \sum_{j} (-i)^k (-)^{2(\frac{k}{2}-j)} \chi_j(\tau) \bar{\chi}_{\frac{k}{2}-j}(\tau) \\
&= (-)^k Z_{(h,h)}(\tau),
\end{aligned}$$

# Two different lines inserted



# SU(N)_k WZW

$$Z_{(h,h)}(\tau) = \sum_{\hat{\mu}\in P_+^k} e^{-\pi i k|A\hat{\omega}_0|^2 - 2\pi i(A\hat{\omega}_0,\hat{\mu})} \bar{\chi}_{A\hat{\mu}}(\bar{\tau})\chi_{\hat{\mu}}(\tau)$$

$$\begin{aligned}
SZ_{(h,h)}(\tau) &= S\sum_{\hat{\mu}\in P_+^k} e^{-\pi i k|A\hat{\omega}_0|^2 - 2\pi i(A\hat{\omega}_0,\hat{\mu})} \bar{\chi}_{A\hat{\mu}}(\bar{\tau})\chi_{\hat{\mu}}(\tau) \\
&= \sum_{\hat{\mu},\hat{\nu}_1,\hat{\nu}_2\in P_+^k} e^{-\pi i k|A\hat{\omega}_0|^2 - 2\pi i(A\hat{\omega}_0,\hat{\mu})} S^*_{\hat{\nu}_1,A\hat{\mu}}\bar{\chi}_{\hat{\nu}_1}(\bar{\tau})S_{\hat{\nu}_2,\hat{\mu}}\chi_{\hat{\nu}_2}(\tau) \\
&= \sum_{\hat{\mu},\hat{\nu}_1,\hat{\nu}_2\in P_+^k} e^{-\pi i k|A\hat{\omega}_0|^2 + 2\pi i(A\hat{\omega}_0,\hat{\nu}_1)} S^*_{\hat{\nu}_1,\hat{\mu}}\bar{\chi}_{\hat{\nu}_1}(\bar{\tau})S_{A\hat{\nu}_2,\hat{\mu}}\chi_{\hat{\nu}_2}(\tau) \\
&= \sum_{\hat{\mu}\in P_+^k} e^{-\pi i k|A\hat{\omega}_0|^2 + 2\pi i(A\hat{\omega}_0,A\hat{\mu})} \bar{\chi}_{A\hat{\mu}}(\bar{\tau})\chi_{\hat{\mu}}(\tau).
\end{aligned}$$

# Consistency Conditions I: continued

- When the order of discrete Abelian $G$ is higher than $Z_2$, we have to truncate the S-transformed twisted partition function to that including primaries operators corresponding to symmetry lines (Verlinde lines).

- We interpret the violation of consistency condition I as a signal of a mixed anomaly for RCFT

- This anomaly agrees precisely with 3d 1-form anomaly [Hung-Wu-Zhou]

# Remarks:

- In diagonal RCFTs, primaries are 1-1 corresponding to Verlinde lines, among which there are symmetry lines we concern. [Verlinde, Petkova-Zuber, Moore-Seiberg]

- Our ‘t Hooft anomaly is a mixed anomaly between global symmetry G and the outer automorphism symmetry. Which is different from F-symbol ‘t Hooft anomaly.

Table 14.1. Outer automorphisms of affine Lie algebras

| g | $\mathcal{O}(\hat{g})$ | Action of the $\mathcal{O}(\hat{g})$ generators |
|---|---|---|
| $A_r$ | $\mathbb{Z}_{r+1}$ | $a[\lambda_0, \lambda_1, \cdots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_0, \cdots, \lambda_{r-2}, \lambda_{r-1}]$ |
| $B_r$ | $\mathbb{Z}_2$ | $a[\lambda_0, \lambda_1, \cdots, \lambda_{r-1}, \lambda_r] = [\lambda_1, \lambda_0, \cdots, \lambda_{r-1}, \lambda_r]$ |
| $C_r$ | $\mathbb{Z}_2$ | $a[\lambda_0, \lambda_1, \cdots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_{r-1}, \cdots, \lambda_1, \lambda_0]$ |
| $D_{r=2\ell}$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $a[\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \lambda_r] = [\lambda_1, \lambda_0, \lambda_2, \cdots, \lambda_r, \lambda_{r-1}]$ |
| | | $\tilde{a}[\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_{r-1}, \lambda_{r-2}, \cdots, \lambda_1, \lambda_0]$ |
| $D_{r=2\ell+1}$ | $\mathbb{Z}_4$ | $a[\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \lambda_r] = [\lambda_{r-1}, \lambda_r, \lambda_{r-2}, \cdots, \lambda_1, \lambda_0]$ |
| $E_6$ | $\mathbb{Z}_3$ | $a[\lambda_0, \lambda_1, \cdots, \lambda_6] = [\lambda_1, \lambda_5, \lambda_4, \lambda_3, \lambda_6, \lambda_0, \lambda_2]$ |
| $E_7$ | $\mathbb{Z}_2$ | $a[\lambda_0, \lambda_1, \cdots, \lambda_7] = [\lambda_6, \lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0, \lambda_7]$ |

# Tests:

- Ising CFT: (Kramers-Wannier duality=anomaly free)
- SU(3)_1:

$$Z_{diag} = |\chi_{\mathbf{1}}|^2 + |\chi_{\mathbf{3}}|^2 + |\chi_{\bar{\mathbf{3}}}|^2. \qquad \widetilde{Z}^{\mathbb{Z}_3} = |\chi_1|^2 + \bar{\chi}_{\bar{3}}\chi_3 + \bar{\chi}_3\chi_{\bar{3}}$$

- SU(3)_2:

$$\widetilde{Z}^{\mathbb{Z}_3} = |\chi_{[2;0,0]}|^2 + \bar{\chi}_{[1;0,1]}\chi_{[1;1,0]} + \bar{\chi}_{[1;1,0]}\chi_{[1;0,1]} + \bar{\chi}_{[0;0,2]}\chi_{[0;2,0]} + |\chi_{[0;1,1]}|^2 + \bar{\chi}_{[0;2,0]}\chi_{[0;0,2]}$$

- SU(3)_3:

$$\widetilde{Z}^{\mathbb{Z}_3} = |\chi_{[3;0,0]} + \chi_{[0;3,0]} + \chi_{[0;0,3]}|^2 + 3|\chi_{[1;1,1]}|^2$$

# Consistency Conditions II

- We have used modular S-transformation, but not yet T.

- If **2** circles on torus inserted by a generating line of $Z_n$, it is expected, after n times of T-twist [CFT yellow book]

$$T^n Z_{(1,h)}(\tau) = Z_{(h^n,h)}(\tau) = Z_{(1,h)}(\tau)$$

- In general, when **2** circles inserted by different lines,

$$Z_{(h^n,h')}(\tau) = Z_{(1,h')}(\tau)$$

# Consistency Conditions II

- The violation of this condition is interpreted as a mixed anomaly between $G$ and diffeomorphism [Numasawa-Yamaguchi]

- This consistency was considered as (generalized) orbifolding conditions when $G=Z_N$ in early days [CFT yellow book]

- We generalize this consistency condition to arbitrary abelian discrete symmetry $G= Z_M*Z_N\ldots$, and find consistency condition I is sufficient but not necessary condition for II.

# WZW Models (Condition I, ‘t Hooft anomaly)

| type | center $\Gamma$ | $S$matrix | $SZ_{(h,h')}\Big\vert = Z_{(h',h)}\Big\vert$ | $\vert A\hat{\mu}\rangle_c = \vert\hat{\mu}\rangle_c$ |
|---|---|---|---|---|
| $A_r$ | $\mathbb{Z}_{r+1}$ | $k \in (r+1)\mathbb{Z}$ | $k \in (r+1)\mathbb{Z}$ | $k \in (r+1)\mathbb{Z}$ |
| $B_r$ | $\mathbb{Z}_2$ | $k \in \mathbb{Z}$ | $k \in \mathbb{Z}$ | $k \in \mathbb{Z}$ |
| $C_r$ | $\mathbb{Z}_2$ | $rk \in 2\mathbb{Z}$ | $rk \in 2\mathbb{Z}$ | $rk \in 2\mathbb{Z}$ |
| $D_{2l}$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $k \in 2\mathbb{Z}$ | $k \in 2\mathbb{Z}$ | $k \in 2\mathbb{Z}$ |
| $D_{2l+1}$ | $\mathbb{Z}_4$ | $k \in 4\mathbb{Z}$ | $k \in 4\mathbb{Z}$ | $k \in 4\mathbb{Z}$ |
| $E_6$ | $\mathbb{Z}_3$ | $k \in 3\mathbb{Z}$ | $k \in 3\mathbb{Z}$ | $k \in 3\mathbb{Z}$ |
| $E_7$ | $\mathbb{Z}_2$ | $k \in 2\mathbb{Z}$ | $k \in 2\mathbb{Z}$ | $k \in 2\mathbb{Z}$ |

[Kikuchi-YZ]

# WZW(Condition II, orbifolding)

| Cartan matrix | Group $G$ | center $\Gamma$ | $\lvert A\hat{\omega}_0\rvert^2$ | $e^{-\pi i N \lvert A\hat{\omega}_0\rvert^2}$ | Anomaly Free |
|---|---|---|---|---|---|
| $A_{n-1}$ | $\mathrm{SU}(n)$ | $\mathbb{Z}_n$ | $\lvert\omega_1\rvert^2 = \frac{n-1}{n}$ | $(-1)^{n-1}$ | $n \in 2\mathbb{Z}+1$ or $k \in 2\mathbb{Z}$ |
| $B_n$ | $\mathrm{Spin}(2n+1)$ | $\mathbb{Z}_2$ | $\lvert\omega_1\rvert^2 = 1$ | $1$ | $k \in \mathbb{Z}$ |
| $C_n$ | $\mathrm{USp}(n)$ | $\mathbb{Z}_2$ | $\lvert\omega_n\rvert^2 = \frac{n}{2}$ | $(-1)^n$ | $n \in 2\mathbb{Z}$ or $k \in 2\mathbb{Z}$ |
| $D_{2l+1}$ | $\mathrm{Spin}(4l+2)$ | $\mathbb{Z}_4$ | $\lvert\omega_1\rvert^2 = \frac{2l+1}{2}$ | $-1$ | $k \in 2\mathbb{Z}$ |
| $E_6$ | $E_6$ | $\mathbb{Z}_3$ | $\lvert\omega_5\rvert^2 = \frac{4}{3}$ | $1$ | $k \in \mathbb{Z}$ |
| $E_7$ | $E_7$ | $\mathbb{Z}_2$ | $\lvert\omega_6\rvert^2 = \frac{3}{2}$ | $-1$ | $k \in 2\mathbb{Z}$ |

$$T^n Z_{(1,h)}(\tau) = \boxed{Z_{(h^n,h)}(\tau) = Z_{(1,h)}(\tau)}$$ [Numasawa-Yamaguchi]

$$\boxed{Z_{(h^n,h')}(\tau) = Z_{(1,h')}(\tau)}$$ [Kikuchi-YZ]

# Boundary state

- Conformal invariance finds Ishibashi states, $|\hat{\mu}\rangle\rangle$

- Physical conditions find Cardy states,

$$|\hat{\mu}\rangle_c := \sum_{\hat{\lambda}\in P_+^k} \frac{S_{\hat{\mu}\hat{\lambda}}}{\sqrt{S_{\hat{0}\hat{\lambda}}}} |\hat{\lambda}\rangle\rangle$$

- For the center symmetry of WZW, it is isomorphic to ``permutation'' of primaries

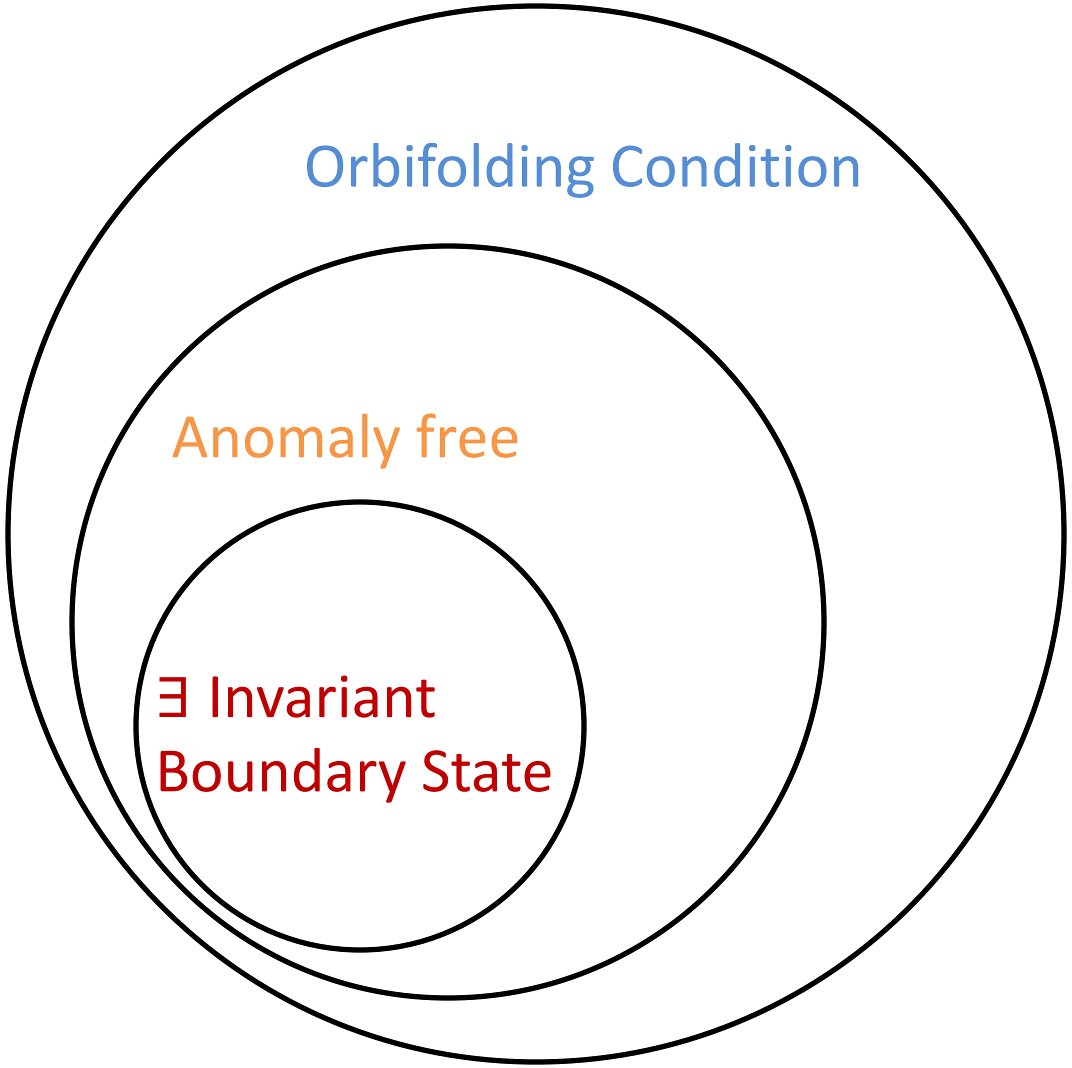
$$h : |\hat{\mu}\rangle_c \mapsto |A\hat{\mu}\rangle_c$$

- Symmetry invariant boundary state is given by

$$|A\hat{\mu}\rangle_c = |\hat{\mu}\rangle_c$$

# $\exists$ invariant boundary state=G is anomaly decoupled

- Invariant boundary state can be used to detect anomaly [Han-Tiwari-Hsieh-Ryu]
- G is anomaly decoupled if G is anomaly (Type I consistency) free and also free of any mixing with other symmetries G'
- For the center symmetry of WZW, we demonstrate the equivalence : G-invariant boundary state = G anomaly decoupled.
- We conjecture this is true for any diagonal RCFT


Orbifolding Condition
Anomaly free
∃ Invariant
Boundary State

# Conclusion

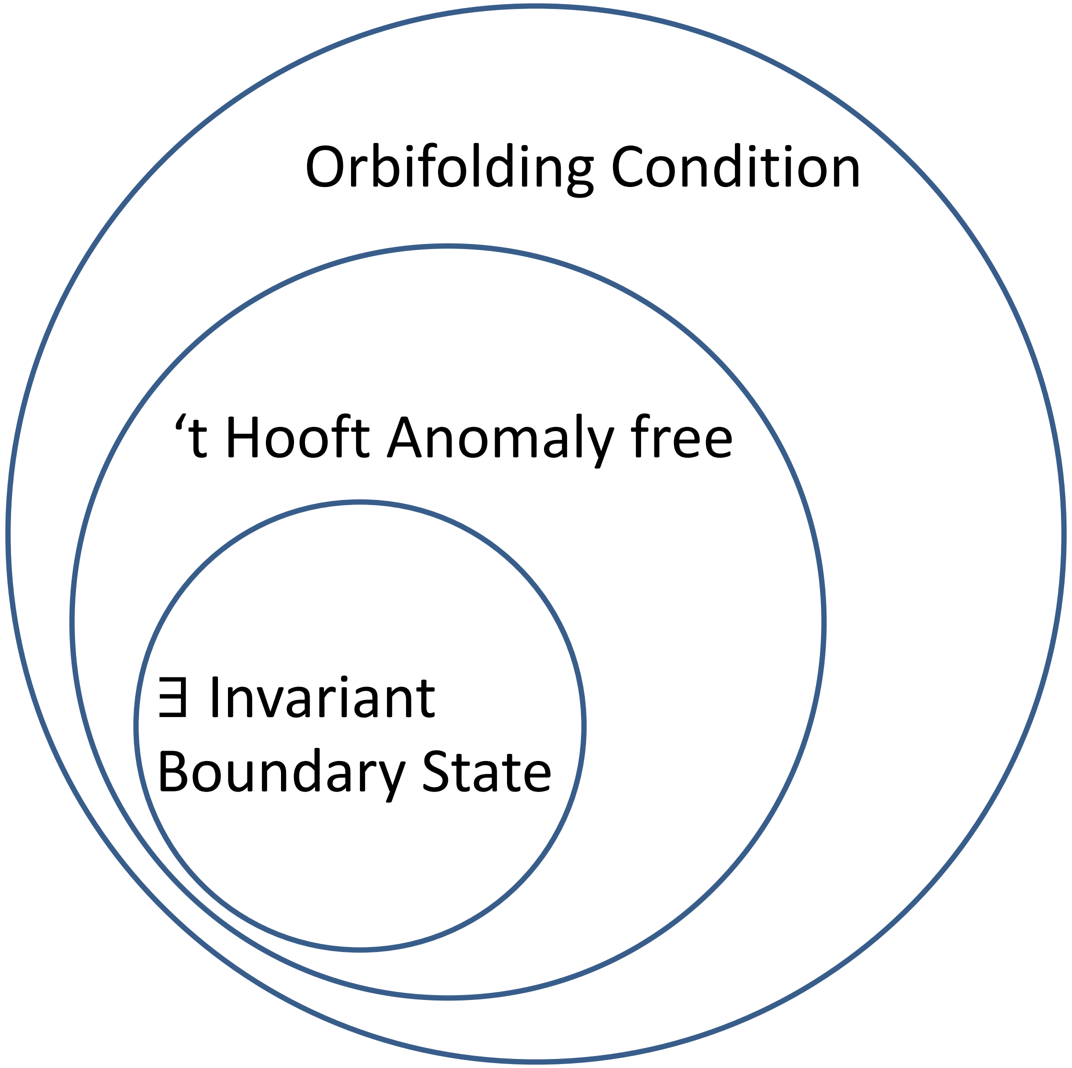
- We find a general way to detect ‘t Hooft anomaly based on twisted torus partition function

- In particular a new anomaly between G and the outer automorphism->such theory can not be trivially gapped

- gapless approach to detect bulk topological phase

- Generalizations? (arbitrary CFT, non-abelian G, higher dimensions, higher form symmetry)

# Thank You!


Orbifolding Condition
‘t Hooft Anomaly free
∃ Invariant
Boundary State