AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities

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Investigation of black hole solutions in Einstein gravity extended with higher-order curvature terms. We consider Einstein-Weyl gravity, with $\mathcal{L} = \sqrt{-g}(R - 2\Lambda + \frac{1}{2}\alpha C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma})$, in four dimensions. As well as all Einstein metrics, there are further non-Einstein solutions, which include additional black holes with AdS or Lifshitz asymptotics.

Based on work with Hong Lü, Yi Pang and Justin Vazquez-Poritz.

AdS/CFT and Other Correspondences

- The AdS/CFT correspondence relates a bulk asymptotically anti de Sitter (AdS) bulk theory with gravity in $d+1$ dimensions to a conformal field theory on its d dimensional Minkowski boundary at infinity.

$$ds^2 = \frac{dr^2}{r^2} + r^2\,\eta_{\mu\nu} dx^\mu dx^\nu$$

  Correlation functions in the strongly-coupled boundary theory at $r \to \infty$ are related to computations involving the classical action in the bulk.

- Non-relativistic theories at strong coupling may be related to analogous bulk backgrounds that are asymptotic to spacetimes where d-dimensional Minkowski symmetry group is broken. Examples include asymptotically Lifshitz spacetimes

$$ds^2 = \frac{dr^2}{r^2} + r^2\,d\vec{x}^2 - r^{2z}\,dt^2$$

  with $z \neq 1$, and asymptotically Schrödinger spacetimes

$$ds^2 = \frac{dr^2}{r^2} + r^2\,(d\vec{x}^2 - 2dvdt) - r^{2z}\,dt^2$$

  These have applications in areas such as condensed matter physics.

Four-Dimensional Higher-Derivative Gravity

- One situation where asymptotically Lifshitz and Schrödinger spacetimes naturally arise is in higher-derivative gravity.

- Higher-derivative gravity may circumvent the non-renormalisability problems of Einstein gravity. Stelle showed that four-dimensional gravity extended with curvature-squared terms is perturbatively renormalisable. At the price, however, of ghosts:

$$-\frac{m^2}{\Box(\Box - m^2)} = \frac{1}{\Box} - \frac{1}{\Box - m^2}$$

- In four dimensions, the Gauss-Bonnet combination is purely topological (total derivative), so the most general possibility with quadratic curvature is

$$I = \frac{1}{2\kappa^2}\int \sqrt{-g}\, d^4x (R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2)$$

- For generic α and β, describes a massless spin-2 field, a massive spin-2 field, and a massive scalar field. Around AdS_4, massless spin-2 and scalar have positive energy excitations, but the massive spin-2 has negative energy (i.e. ghostlike).

Einstein-Weyl Gravity

- The massive spin-0 is absent if $\alpha = -3\beta$. The action is just

$$I = \frac{1}{2\kappa^2} \int \sqrt{-g}\, d^4x \Big(R - 2\Lambda + \tfrac{1}{2}\alpha C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \Big)$$

  where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

- The equations of motion are $\mathcal{G}_{\mu\nu} - 4\alpha E_{\mu\nu} = 0$, where

$$\begin{aligned} \mathcal{G}_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu}\,, \\ E_{\mu\nu} &= (\nabla^\rho \nabla^\sigma + \tfrac{1}{2} R^{\rho\sigma}) C_{\mu\rho\nu\sigma} \end{aligned}$$

  and $E_{\mu\nu}$ is the Bach tensor.

- The Bach tensor vanishes if $g_{\mu\nu}$ is Einstein, so all solutions of cosmological Einstein gravity remain solutions. We can consider the AdS$_4$ background, with

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})\,, \qquad R_{\mu\nu} = \Lambda g_{\mu\nu}$$

  and then study linearised fluctuations around this. When needed, we shall take the AdS$_4$ metric to be

$$ds^2 = \frac{3}{(-\Lambda)} \Big[-\cosh^2 \rho\, dt^2 + d\rho^2 + \sinh^2 \rho\, d\Omega_2^2 \Big]$$

Linearisation Around AdS$_4$

- The metric fluctuations $\delta g_{\mu\nu} = h_{\mu\nu}$ around AdS$_4$ then satisfy

$$[\alpha\Box + 1 - \tfrac{4}{3}\alpha\Lambda]\mathcal{G}^L_{\mu\nu} - \tfrac{2}{3}\alpha\Lambda R^L g_{\mu\nu}$$
$$-\tfrac{1}{3}\alpha(-\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box + \Lambda g_{\mu\nu})R^L = 0$$

where the linearised Einstein and Ricci tensors are given by

$$\begin{aligned}
\mathcal{G}^L_{\mu\nu} &= R^L_{\mu\nu} - \tfrac{1}{2}R^L\, g_{\mu\nu} - \Lambda\, h_{\mu\nu}\,, \\
R^L_{\mu\nu} &= \nabla^\lambda\nabla_{(\mu}h_{\nu)\,\lambda} - \tfrac{1}{2}\Box h_{\mu\nu} - \tfrac{1}{2}\nabla_\mu\nabla_\nu h\,, \\
R^L &= \nabla^\mu\nabla^\nu h_{\mu\nu} - \Box h - \Lambda h\,.
\end{aligned}$$

- A convenient gauge choice here is $\nabla^\mu h_{\mu\nu} = \nabla_\nu h$, where $h = g^{\mu\nu}h_{\mu\nu}$. (Similar to de Donder gauge, except 1 rather than $\frac{1}{2}$ factor on RHS.) The trace of the fluctuation equation then gives

$$\Lambda h = 0$$

This then means $\nabla^\mu h_{\mu\nu} = 0$, so $h_{\mu\nu}$ is transverse, traceless. Setting $\Lambda = -3$ ("unit AdS$_4$") for convenience, the modes satisfy

$$\Big(\Box + 2\Big)\Big(\Box + 4 + \frac{1}{\alpha}\Big)\, h_{\mu\nu} = 0$$

The Spin-2 Modes

- Generically, when the constant terms in the two factors are unequal, the general solution of this fourth-order equation is a linear combination of solutions of the two factorised equations

$$(\Box + 2)h^{(0)}_{\mu\nu} = 0 \quad \text{and} \quad \left(\Box + 4 + \frac{1}{\alpha}\right) h^{(M)}_{\mu\nu} = 0$$

- $h^{(0)}_{\mu\nu}$ is an ordinary massless spin-2 field, and $h^{(M)}_{\mu\nu}$ describes massive spin-2, with $M^2 = -2 - 1/\alpha$.

- Irreducible representations $D(E_0, s)$ of the AdS group $SO(3,2)$ are characterised by their lowest energy E_0 and their spin s. These are the eigenvalues of the representation under L_{04} and L_{12}, where L_{AB} are the $SO(3,2)$ generators. Unitarity requires $E_0 \geq s + 1$, with masslessness in the case of equality.

- For spin-2, E_0 is related to M by

$$E_0 = \frac{3}{2} \pm \sqrt{\frac{9}{4} + M^2}$$

  Thus unitary spin-2 requires the plus sign, and $M^2 \geq 0$.

- Can have non-tachyonic negative M^2, provided $-\frac{9}{4} \leq M^2 < 0$.

Energies of the spin-2 Modes

- The energies of the massless and massive spin-2 modes, calculated from the Hamiltonian describing small fluctuations around AdS_4, are

$$E^{(0)} = -\frac{1}{2\kappa^2}(1+2\alpha)\int\sqrt{-g}\,d^4x\,\nabla^0 h^{\mu\nu}_{(0)}\,\dot{h}^{(0)}_{\mu\nu}$$
$$E^{(M)} = \frac{1}{2\kappa^2}(1+2\alpha)\int\sqrt{-g}\,d^4x\,\nabla^0 h^{\mu\nu}_{(M)}\,\dot{h}^{(M)}_{\mu\nu}$$

  The integrals are negative, so we need

$$\alpha \ge -\frac{1}{2}$$

  in order to avoid negative energies for the massless spin-2 modes. The massive spin-2 then have negative energies.

- One way to avoid the massive spin-2 ghosts could be to eliminate these modes by imposing suitable boundary conditions at infinity. The fall-off is of order $e^{-E_0\,\rho}$, so lower E_0 modes fall off more slowly. In particular, the non-unitary modes with $E_0 < 3$ fall off more slowly than the massless modes.

- If we arrange for all the massive spin-2 to have $E_0 < 3$, then boundary conditions could eliminate the ghosts. (This is effectively done anyway for the minus-sign choice in E_0.)

The Non-Tachyonic Window

- The massive modes are non-unitary and still non-tachyonic if $-\frac{9}{4} \le M^2 < 0$. This translates into

$$\alpha \le -\frac{1}{2} \qquad \text{or} \qquad \alpha \ge 4$$

- We need $\alpha \ge -1/2$ for the massless modes to be ghostfree, so this leaves

$$\alpha = -\frac{1}{2} \qquad \text{or} \qquad \alpha \ge 4$$

- The case $\alpha = -1/2$ is *Critical Gravity*. Here the massless modes have zero energy, and the massive modes become massless , giving logarithmic modes (also non-unitary).

- $\alpha \ge 4$ gives a 1-parameter family of models where the massless modes have positive energies, and the massive modes are all non-unitary, $E_0 < 3$, and can be eliminated by boundary conditions at infinity.

- The limiting case $\alpha \to \infty$ is pure Weyl-squared conformally-invariant gravity, as discussed recently by Maldacena.

Lifshitz and Schrödinger Vacua

- In addition to AdS, Einstein-Weyl gravity also admits Lifshitz vacua,

$$ds^2 = -r^{2z}\, dt^2 + \frac{dr^2}{\sigma r^2} + r^2(dx^2 + dy^2)$$

with

$$z = \frac{8\alpha + 1 \pm \sqrt{2(1+2\alpha)(16\alpha - 1)}}{4\alpha - 1}\,, \qquad \sigma = \frac{6}{z^2 + 2z + 3}$$

z is real if $\alpha \ge 1/16$ or $\alpha \le -1/2$.

- In conformal gravity ($\alpha \longrightarrow \infty$), the Lifshitz scaling is $z = 4$ or $z = 0$. These solutions generalise in conformal gravity to S^2 or H^2 spatial topologies.

- There exist also Schrödinger vacua in Einstein-Weyl gravity,

$$ds^2 = -r^{2z}\, dt^2 + \frac{dr^2}{r^2} + r^2(-2dt\, dx + dy^2)$$

with

$$z = \frac{1}{4} \pm \frac{1}{4}\sqrt{1 - \frac{4}{\alpha}}$$

Solutions in Einstein-Weyl and Conformal Gravity

- Since the Bach tensor vanishes for Einstein metrics, all solutions of cosmological Einstein gravity $\mathcal{L} = \sqrt{-g}(R - 2\Lambda)$ are also solutions of Einstein-Weyl gravity (with the same Λ). In particular, Schwarzschild-AdS and Kerr-AdS are solutions.

- Beyond the Einstein metrics, it is difficult to obtain general classes of explicit solutions of the fourth-order equations of Einstein-Weyl gravity, even if one assumes spherical symmetry (à la Schwarzschild-AdS). More can be done explicitly in pure conformal gravity:

- Since the equations of motion of conformal gravity are conformally invariant, **any metric that is conformally Einstein is a solution**.

- In fact, solutions of conformal gravity that are *not* conformally Einstein are quite hard to come by.

- A test for being conformally Einstein: If $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $\tilde{R}_{\mu\nu} = \frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu}$, then $V_\mu \equiv \partial_\mu \log\Omega$ satisfies

$$\nabla^\mu C_{\mu\nu\rho\sigma} + V^\mu C_{\mu\nu\rho\sigma} = 0$$

So the existence of such a V^μ is a necessary condition.

Non-conformally Einstein Solutions of Conformal Gravity

- As of 2000 (2008?), only one Lorentzian solution of conformal gravity that is not conformally Einstein was known (Nurowski and Plebanski):

$$ds^2 = dx^2 + dy^2 - \tfrac{2}{3}(dx + y^3 du)(ydr + \tfrac{11}{9}dx - \tfrac{1}{9}y^3 du)$$

  (And any conformal scaling thereof.)

- In Euclidean signature, any metric with (anti) self-dual Weyl tensor,

$$C_{\mu\nu\rho\sigma} = \pm\tfrac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}C_{\alpha\beta\rho\sigma}$$

  is a solution of conformal gravity. Non-conformally Einstein examples are known.

- Among Bianchi IX metrics,

$$ds^2 = dr^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

  where a, b and c are functions of r and σ_i are left-invariant 1-forms of $SU(2)$, there exist triaxial solutions ($a \neq b \neq c \neq a$) of conformal gravity that are not conformally Einstein. (All biaxial solutions ($a = b$) *are* conformally Einstein.)

Spherically-Symmetric Black Holes in Conformal Gravity

- Using general coordinate invariance and conformal scaling, the most general conformal class of "spherically symmetric" metrics is

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega_{2,k}^2\,,$$

where

$$d\Omega_{2,k}^2 = \frac{dx^2}{1-kx^2} + (1-kx^2)dy^2$$

with $k = 1$, 0 or -1 (S^2, T^2 or H^2). The metric solves the conformal gravity equations if (Reigert,...)

$$f = br^2 + \frac{c^2-k^2}{3d}\,r + c + \frac{d}{r}$$

where b, c and d are constants.

- If $c^2 = k^2$, the metric is Schwarzschild-AdS. When $c^2 \neq k^2$, there is an additional parameter, associated with a slower fall-off (relative to the br^2 cosmological term) than the usual "mass term" d/r. This may be viewed as massive spin-2 "hair."

Black Hole Thermodynamics in Conformal Gravity

- The general spherically symmetric solution in conformal gravity is in fact a conformal scaling of Schwarzschild-AdS $d\tilde{s}^2$, namely $ds^2 = \Omega^{-2}\, d\tilde{s}^2$ with

  $$d\tilde{s}^2 = -\Big(k - \tfrac{1}{3}\Lambda\rho^2 - \frac{2M}{\rho}\Big)dt^2 + \Big(k - \tfrac{1}{3}\Lambda\rho^2 - \frac{2M}{\rho}\Big)^{-1} d\rho^2 + \rho^2 d\Omega_{2,k}^2$$

  with $\Omega = 1 + q\rho$ and $r = \rho\Omega^{-1}$. The parameters in the general conformal gravity solution are then given by

  $$b = 2Mq^3 + kq^2 - \tfrac{1}{3}\Lambda\,, \qquad c = k + 6Mq\,, \qquad d = -2M$$

  Since the conformal factor is singular at $\rho = \infty$, the thermodynamic properties of the black hole in conformal gravity differ from those of the Schwarzschild-AdS black hole.

- Calculation of the mass of the general black hole in conformal gravity requires care. The presence of the slower fall-off of the extra "massive spin-2" term in the metric function f means that the usual methods for calculating the mass give divergent results. (Deser-Tekin involves a background subtraction, which is often fraught with ambiguities. Even AMD (electric component of Weyl tensor at infinity) diverges because of slower approach to AdS.)

- For any vector ξ, first variation of Lagrangian 4-form is given by $\mathcal{L}_\xi L = \mathcal{E}\mathcal{L}_\xi\phi + d\Theta(\phi, \mathcal{L}_\xi\phi)$, where ϕ represents the fields (metric,...) and $\mathcal{E}$ the equations of motion. Then if ξ generates a symmetry, there is a conserved current $J = \Theta - i_\xi L$ with $J = dQ$ on-shell. For conformal gravity,

$$Q = \frac{\alpha}{4}\epsilon_{\mu\nu\rho\sigma}\,(C^{\rho\sigma\alpha\beta}\nabla_\alpha\xi_\beta - 2\xi_\beta\nabla_\alpha C^{\rho\sigma\alpha\beta})dx^\mu \wedge dx^\nu$$

  This gives a finite result for the mass E of the general conformal black hole, and it agrees with the standard formula when $c^2 = k^2$.

- If we write the metric function as

$$f = -\tfrac{1}{3}\Lambda r^2 + \Xi r + c + \frac{d}{r}$$

  where $3\Xi d = c^2 - k^2$, then the mass E satisfies the first-law relation

$$dE = TdS + \Theta d\Lambda + \Psi d\Xi$$

  and the Smarr-type formula $E = 2\Theta\Lambda + \Psi\Xi$, where

$$\Psi = \frac{\alpha(c-k)}{24\pi}$$

  is the thermodynamic variable conjugate to the massive hair parameter Ξ.

$z = 4$ Lifshitz Black Holes in Conformal Gravity

- These are given by

$$ds^2 = -r^8 f dt^2 + \frac{4dr^2}{r^2 f} + r^2 d\Omega_{2.k}^2\,, \qquad f = 1 + \frac{c}{r^2} + \frac{c^2 - k^2}{3r^4} + \frac{d}{r^6}$$

  The metric is again conformal to Schwarzschild-AdS $d\tilde{s}^2 = \Omega^2 ds^2$ with $\Omega = q[r(c-k+3r^2)]^{-1}$ and $\rho = r\Omega$. The conformal factor is singular at infinity, and so the asymptotic global structure is very different from Schwarzschild-AdS.

- It is not obvious how to define, or calculate, the black hole mass in this Lifshitz case. Not only the usual Deser-Tekin and AMD methods, but also the conserved charge Q we used earlier, give divergent answers. One possibility is to defined the mass E by integrating the first law $dE = TdS$.

- For $k = 0$ solutions, there exists a conserved Noether charge associated with a global scaling symmetry of the metrics (analogous to one considered by Bertoldi, Burrington and Peet for Lifshitz black holes in Einstein-Proca). This can be used to give a definition of mass in these cases, which is in agreement with the integration of the first law.

- There is also a $z = 0$ Lifshitz black hole, with $ds^2 = -f dt^2 + 4dr^2/(r^2 f) + r^2 d\Omega_{2,k}^2$ and $f = 1 + c/r^2 + (c^2 - k^2)/(3r^4)$.

AdS and Lifshitz Black Holes in Einstein-Weyl Gravity

- We no longer have conformal symmetry, and the general ansatz for spherically-symmetric solutions is

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,k}^2$$

The equations can be reduced to second-order equations for $a(r)$ and $f(r)$, but these seem not to be solvable explicitly.

- We can use numerical methods to solve the equations. A convenient way is to construct a series expansion near the horizon at $r = r_0$, and use this to set initial conditions just outside the horizon.

$$\begin{aligned} a(r) &= (r - r_0) + a_2(r - r_0)^2 + a_3(r - r_0)^3 + \cdots \\ f(r) &= f_1(r - r_0) + f_2(r - r_0)^2 + f_3(r - r_0)^3 + \cdots \end{aligned}$$

a_n and f_n for $n \geq 2$ are solved for in terms of r_0 and f_1.

- Schwarzschild-AdS corresponds to $f_1 = 3r_0 + k/r_0$. Defining

$$f_1 = 3r_0 + \frac{k}{r_0} + \delta\,,$$

then taking $\delta \neq 0$ corresponds to turning on the massive spin-2 "hair."

- We find that if the Weyl-squared coefficient α lies in the range where the massive spin-2 excitations around AdS have $m^2 < 0$, namely $\alpha < -\frac{1}{2}$ or $\alpha > 0$, then there exists a range of δ around $\delta = 0$ for which the equations of motion integrate stably out to $r = \infty$.

$$\delta_- \le \delta \le \delta_+ \,.$$

- For δ *inside* this range, the solution approaches AdS as $r \to \infty$.

- For $\delta = \delta_-$ or $\delta = \delta_+$, the solution instead approaches the Lifshitz vacuum discussed previously, with z being the larger root of

$$z = \frac{8\alpha + 1 \pm \sqrt{2(1 + 2\alpha)(16\alpha - 1)}}{4\alpha - 1}$$

- If δ outside the range, the solution is singular at large r.

- Thus when $\alpha < -\frac{1}{2}$ or $\alpha > 0$, there exists a 1-parameter family of more general asymptotically AdS spherically symmetric black holes with massive spin-2 "hair." There exist also two discrete endpoints of the parameter range where the black hole is asymptotically Lifshitz.

- If $0 < \alpha < 4$, the massive mode has $m^2 < -9/4$, outside the Breitenlöhner-Freedman bound. Then, the solutions will be unstable to time-dependent runaway modes.

Conclusions

- Einstein-Weyl gravity with $\alpha < -\frac{1}{2}$ or ≥ 4 provides a possibly viable description of gravity in which the massive spin-2 modes, which have slower fall-off than the massless modes, could be truncated by boundary conditions. (Similar to the mechanism discussed by Maldacena for conformal gravity.)

- All solutions of Einstein gravity are also solutions of Einstein-Weyl gravity. Finding the more general solutions of Einstein-Weyl gravity explicitly is difficult, even for spherical symmetry. Numerical analysis indicates further black holes exist, both with AdS and Lifshitz asymptotics.

- The equations of motion of pure conformal gravity ($\alpha \to \infty$) are easier to solve. There are interesting open problems here, such as finding classes of solutions that are not conformally Einstein.

- The thermodynamics of the black holes in conformal and in Einstein-Weyl gravity is not well understood. Can one give a meaning to energy for asymptotically Lifshitz black holes?

- It would be interesting to study more complicated solutions, such as rotating black holes.