Recent Progress in **Stochastic Gravity**:

Stress-energy Tensor Fluctuations

in Early Universe Quantum Processes and for Black Holes Backreaction Problems

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Semiclassical Gravity

Semiclassical Einstein Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle\hat{T}_{\mu\nu}\rangle_q \quad + \kappa\,(T_{\mu\nu})_c$$

$\tilde{G}_{\mu\nu}$ is the Einstein tensor (plus covariant terms associated with the renormalization of the quantum field)

$\kappa = 8\pi G_N$ and G_N is Newton's constant

Free massive scalar field

$$(\Box - m^2 - \xi R)\hat{\phi} = 0.$$

$\hat{T}_{\mu\nu}$ is the stress-energy tensor operator

$\langle\,\rangle_q$ denotes the expectation value

Part I: Stochastic Gravity

Einstein- Langevin Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa \left(T^{\mathrm{c}}_{\mu\nu} + T^{\mathrm{qs}}_{\mu\nu} \right)$$

$T^{c}_{\mu\nu}$ is due to classical matter or fields

$$T^{\mathrm{qs}}_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle_{\mathrm{q}} + T^{\mathrm{s}}_{\mu\nu}$$

$T^{\mathrm{qs}}_{\mu\nu}$ is a new stochastic term
related to the quantum fluctuations of $T_{\mu\nu}$

NOISE KERNEL

- Exp Value of 2-point correlations of stress tensor: bitensor
- Noise kernel measures quantum flucts of stress tensor

$$N_{abcd}(x,y) = \frac{1}{2}\langle\{\hat{t}_{ab}(x), \hat{t}_{cd}(y)\}\rangle$$

$$\hat{t}_{ab} \equiv \hat{T}_{ab} - \langle \hat{T}_{ab} \rangle \hat{I}$$

- The noise kernel is real and positive semi-definite as a consequence of stress energy tensor being self-adjoint

the ultraviolet behaviour of $\langle \hat{T}_{ab}(x)\hat{T}_{cd}(y)\rangle$ is the same as that of $\langle \hat{T}_{ab}(x)\rangle\langle \hat{T}_{cd}(y)\rangle$,

How could a quantum field give rise to a stochastic source?

It can be represented by a classical stochastic tensor source

$$\xi_{ab}[g]$$

(Gaussian via influence functional Feynman and Vernon 1963)

$$\langle \xi_{ab} \rangle_s = 0 \qquad \langle \xi_{ab}(x)\xi_{cd}(y) \rangle_s = N_{abcd}(x,y)$$

- **Symmetric, traceless** (for conformal field), **divergenceless**

Classical Stochastic Field assoc.with a Quantum Field

- Stochastic tensor is covariantly conserved in the background spacetime (which is a solution of the semiclassical Einstein equation).

$$\nabla^a \xi_{ab}[g; x) = 0.$$

- For a conformal field ξ_{ab} is traceless:

$$g^{ab}\xi_{ab}[g; x) = 0;$$

Thus there is no stochastic correction to the trace anomaly

Einstein-Langevin Equation

[Hu & Matacz, PRD1994, Campos Verdaguer 1996, Lombardi and Mazzitelli, 96 ..]

- We will assume linear perturbation of semiclassical solution

$$g_{ab} + h_{ab}$$

But stochastic gravity is NOT restricted to linear perturbations

- Einstein-Langevin equation: $G_{g+h} = \kappa(\langle \hat{T} \rangle_{g+h} + \xi)$

$$G^{(1)}_{ab}[g+h] = \kappa \langle \hat{T}^{(1)}_{ab}[g+h] \rangle_{ren} + \kappa \xi_{ab}[g]$$

$$(\nabla^2_{g+h} - m^2 - \xi R)\hat{\phi} = 0$$

Applications of Stochastic Gravity: Fluctuations & Back-reaction Problems

1. Validity of Semiclassical Gravity — Hu, Roura, Verdaguer (PRD04)

- Stability of solutions to SC Einstein Eqn with contributions of fluctuations — Einstein-Langevin Eqn
- Stochastic Gravity as next-to-leading-order 1/N limit. (Roura Verdaguer 03, Hartle-Horowitz 80)

2. Vacuum Fluctuations of Quantum Fields & Induced effects on Spacetime Dynamics:

- Negative energy density, quantum interest (Ford Roman)
- Re-examine Classical theorems in GR: Energy Dominance Condi[tion] with effects of quantum fluctuations

3. Black Hole Horizon Fluctuations & Backreaction (Hu, Raval, Sinha 98, 03)

- many speculations on the magnitude of such fluctuations but no quantitative calculations yet.
- Stochastic gravity is the theory for such inquiries

Roura Hu (06, 07)

4. Structure Formation from grav. perturbations (Roura & Verdaguer 03

- particularly useful for trace anomaly-induced inflations.

Roura & Verdaguer (99, 07), Urakawa and Maeda (07)

5. A platform towards Quantum Gravity —

defined as a theory of the micro-scopic structure of ST

NOT quantizing GR

Hu 03

Noise Kernel from Green Functions

Phillips and Hu, PRD63, 104001 (2001)

- Wightman Function

$$G_{xy} \equiv G_{+}(x,y) = \langle \hat{\phi}(x)\ \hat{\phi}(y) \rangle]$$

$$\langle \hat{\phi}(x)\ \hat{\phi}(y)\ \hat{\phi}(x')\ \hat{\phi}(y') \rangle = G_{xy'}\, G_{yx'} + G_{xx'}\, G_{yy'} + G_{xy}\, G_{x'y'}\,. \qquad (3.15)$$

$$N_{abc'd'}(x,y) = N_{abc'd'}[G_{+}(x,y)] + N_{abc'd'}[G_{+}(y,x)]. \qquad (3.21)$$

For timelike separated points, can express in terms of the Feynman (time ordered) Green function $G_F(x,y)$ and the Dyson (anti-time-ordered) Green function $G_D(x,y)$:

$$N_{abc'd'}(x,y) = N_{abc'd'}[G_F(x,y)] + N_{abc'd'}[G_D(x,y)].^{7} \quad (3.22)$$

Hadamard (or Schwinger) function

$$G^{(1)}(x,x') = \langle \{ \hat{\phi}(x), \hat{\phi}(x') \} \rangle$$

{curly brackets} denote symmetrized operator product.

Ford et al have used the Hadamard function for consideration of effects of stress energy tensor fluctuations

A general expression for the Noise Kernel in terms of four covariant derivatives acting on products of the Green function for a given quantum field

$$
\begin{aligned}
8\tilde{N}_{abc'd'}[G] = {} & (1-2\xi)^2 (G_{;c'b} G_{;d'a} + G_{;c'a} G_{;d'b}) + 4\xi^2 (G_{;c'd'} G_{;ab} + G\, G_{;abc'd'}) \\
& -2\xi(1-2\xi)(G_{;b} G_{;c'ad'} + G_{;a} G_{;c'bd'} + G_{;d'} G_{;abc'} + G_{;c'} G_{;abd'}) \\
& +2\xi(1-2\xi)(G_{;a} G_{;b} R_{c'd'} + G_{;c'} G_{;d'} R_{ab}) - 4\xi^2 (G_{;ab} R_{c'd'} + G_{;c'd'} R_{ab}) G \\
& +2\xi^2 R_{c'd'} R_{ab} G^2, \qquad (3.2
\end{aligned}
$$

$$
\begin{aligned}
8\tilde{N}'_{ab}[G] = {} & 2(1-2\xi)[(2\xi-\tfrac{1}{2}) G_{;p'b} G_{;}{}^{p'}{}_{a} + \xi(G_{;b} G_{;p'a}{}^{p'} + G_{;a} G_{;p'b}{}^{p'})] \\
& -4\xi[(2\xi-\tfrac{1}{2}) G_{;}{}^{p'} G_{;abp'} + \xi(G_{;p'}{}^{p'} G_{;ab} + G\, G_{;abp'}{}^{p'})] - (m^2+\xi R')[(1-2\xi) G_{;a} G_{;b} \\
& -2G\xi G_{;ab}] + 2\xi[(2\xi-\tfrac{1}{2}) G_{;p'} G_{;}{}^{p'} + 2G\xi G_{;p'}{}^{p'}] R_{ab} - (m^2+\xi R')\xi R_{ab} G^2, \qquad (3.2
\end{aligned}
$$

$$
\begin{aligned}
8\tilde{N}[G] = {} & 2(2\xi-\tfrac{1}{2})^2 G_{;p'q} G_{;}{}^{p'q} + 4\xi^2 (G_{;p'}{}^{p'} G_{;q}{}^{q} + G\, G_{;p}{}^{p}{}_{q'}{}^{q'}) \\
& +4\xi(2\xi-\tfrac{1}{2})(G_{;p} G_{;q'}{}^{pq'} + G_{;}{}^{p'} G_{;q}{}^{q}{}_{p'}) - (2\xi-\tfrac{1}{2})[(m^2+\xi R) G_{;p'} G_{;}{}^{p'} + (m^2+\xi R') G_{;p} G^{;p}] \\
& -2\xi[(m^2+\xi R) G_{;p'}{}^{p'} + (m^2+\xi R') G_{;p}{}^{p}] G \tfrac{1}{2}(m^2+\xi R)(m^2+\xi R') G^2. \qquad (3.2
\end{aligned}
$$

Phillips and Hu, PRD63, 104001 (2001)

Essential to obtain expressions for the Noise Kernels -- *Exp Value of 2-point correlations of stress tensor bitensor* for the investigation of fluctuations and backreaction problems

- An important example is **fluctuations in an evaporating black hole** and the
- **Backreaction** of Hawking radiation on it.
- **Black hole end-state** problem and **information loss** issue.

Part II. Black Holes

- A black hole emits **thermal** radiation (fixed background geometry): $T = m_\mathrm{p}^2/8\pi M$ Hawking
- **Back reaction** on the geometry using semiclassical gravity $\Rightarrow$ black hole **evaporation**

$$\frac{dM}{dt} \propto -m_\mathrm{p}^2 \left(\frac{m_\mathrm{p}}{M}\right)^2$$ Bardeen; Massar

- Horizon fluctuations involving long timescales (evaporation time):

  - Evaporating BH: Ford & Wu $\longrightarrow$ $\delta M \sim m_\mathrm{p}$
    Bekenstein $\longrightarrow$ $\delta M \sim M$ when $M \sim (m_\mathrm{p} M_0^2)^{1/3}$

Semiclassical Gravity

- **Semiclassical** Einstein equation:

$$G_{ab}[g] = \kappa \left\langle \hat{T}_{ab}[g] \right\rangle_{\mathrm{ren}} \qquad \kappa = \frac{8\pi}{m_p^2}$$

- Spherically symmetric geometry:

$$ds^2 = -e^{2\psi(v,r)}\left(1 - \frac{2m(v,r)}{r}\right)dv^2 + 2e^{\psi(v,r)}dv\,dr + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Semiclassical Einstein equation: $L_{\mathrm{H}} \propto 1/M^2$

$$\begin{aligned}
\frac{\partial m}{\partial v} &= 4\pi r^2 T^r_v = -e^{\psi} L_{\mathrm{H}} + O\left(m L_{\mathrm{H},v}\right) \\
\frac{\partial m}{\partial r} &= -4\pi r^2 T^v_v = O\left(L_{\mathrm{H}}\right) \\
\frac{\partial \psi}{\partial r} &= 4\pi r T_{rr} = O\left(L_{\mathrm{H}}/r\right)
\end{aligned}$$

- Energy-momentum *conservation:*

$$\frac{\partial\left(r^2 T^r_v\right)}{\partial r} + r^2 \frac{\partial T^v_v}{\partial v} = 0 \qquad 2M \ll r \ll M/L_{\mathrm{H}}$$

Relates *outgoing* <u>positive</u> energy flux at large r to *ingoing* <u>negative</u> flux on the horizon.

Stochastic Gravity

- **Einstein-Langevin** equation:

$$G_{ab}^{(1)}[g+h] = \kappa \left\langle \hat{T}_{ab}^{(1)}[g+h] \right\rangle_{\text{ren}} + \kappa \xi_{ab}$$

$$\langle \xi_{ab}(x) \xi_{cd}(y) \rangle_\xi = \frac{1}{2} \langle \{ \hat{t}_{ab}(x), \hat{t}_{cd}(y) \} \rangle, \quad \hat{t}_{ab} \equiv \hat{T}_{ab} - \langle \hat{T}_{ab} \rangle$$

For spherically symmetric systems:

- $l = 0$ sector of metric fluctuations:
  - not a complete solution,
  - but goes well beyond 2-D dilaton-gravity models.
- Proceed analogously to the mean evolution

Different views, divergent claims, but none from real calculations based on solid viable theories

- Assuming simple connection between *outgoing* and *ingoing* energy flux fluctuations: Ford & Wu

- Simple qualitative explanation: (even without fluctuations)
  Initially $\delta M \sim m_{\mathrm{p}} \longrightarrow \delta M \sim M$ for $M \sim M_0^{2/3}$.

Ford and Wu neglected important

Secular effect due to $\langle \hat{T}^{(1)}_{ab}[g+h] \rangle_{\mathrm{ren}}$.

Our result agreed with Bekenstein 83

B. L. Hu and A. Roura, *Metric fluctuations of an evaporating black hole from back-reaction of stress tensor fluctuations* Phys. Rev. D 76 (2007) 124018

In-out energy flux connected. Yet, **Flux fluctuations not connected**

- The **inequivalence** can be explicitly proved:

$$\int dv\, h(v) \int dv' h(v') r^4 \langle \xi_v^r(v,r) \xi_v^r(v',r) \rangle_\xi$$

divergent on the horizon, but finite away from it.

Main conclusions

from Hu and Roura, *Phys. Rev. D 76 (2007) 124018*

- Large fluctuations build up over **long times** for **evaporating** black holes.
- Assumptions about **correlations** between *outgoing* and *ingoing* not accurate → need to
  - study fluctuations *near the horizon* in detail,
  - find the right way to *probe* those metric fluctuations.

Noise Kernel needed for Fluctuations Phenomena and Backreaction Problems

We describe recent results in the calculations of the noise kernels of conformally-invariant scalar fields in:

0) spacetimes conformal to an ultrastatic spacetime (under quasilocal approximation) A. Eftekharzadeh, Jason Bates, Albert Roura, P. R. Anderson and B. L. Hu, *Noise kernel for a quantum field in Schwarzschild spacetime under the Gaussian approximation,* Phys. Rev. D85, 044037 (2012)

1) static de Sitter and all conformally-flat spacetimes (exact expressions)
* This talk: Jason D. Bates, Hing-Tong Cho, Paul R. Anderson and B. L. Hu, *Exact noise kernel for quantum fields in static de Sitter and conformally-flat spacetimes* (in preparation)

2) massive quantum scalar field with arbitrary coupling in Euclidean R^N and **AdS^N spaces** via the **generalized zeta-function method.**

[Hing-Tong Cho and B. L. Hu, "*Stress-energy Tensor Correlators of a Quantum Field in Euclidean R^N and AdS^N spaces via the generalized zeta-function method*" Phys. Rev. D84, 044032 (2011)]

- Black hole in thermal **equilibrium:**
  - *Stability* → AdS / box (reflecting or isothermal walls)
  - Semiclassical *back reaction* → small correction York

Backreaction with Fluctuations: Program already laid out in S. Sinha, A. Raval and B. L. Hu, "Black Hole Fluctuations and Backreaction in Stochastic Gravity" in Bekenstein issue of Foundations of Physics *Thirty Years of Black Hole Physics* edited by L. Horwitz (2003). [gr-qc/0210013] Summarized in B. L. Hu and E. Verdaguer, **Stochastic gravity: A primer with applications**, Class. Quant. Grav. **20** (2003) R1-R42 [gr-qc/0211090]

Expect fluctuations small, can use thermodynamic arguments or QFT

- Equilibrium BH: Zurek → $\delta M \sim m_{\rm p}$

Strategy for calculating NKs in (quasistatic) black hole spacetimes:

- *Hartle-Hawking* state ⟶ *Minkowski* vacuum.
  Boulware state ⟶ *Rindler* vacuum (large fluctuations near the horizon).
- Near horizon region for arbitrary l ⟶ de Sitter.
  Hartle-Hawking state ⟶ *Bunch-Davies* vacuum.

Go to de Sitter spacetime. Static coordinate:
Gibbons-Hawking state corresponds to Hartle-Hawking state.

Minkowski to conformally-flat

- One step further back, start with **Minkowski,** then to conformally-related spacetimes.

  NK in **Minkowski** space:

  R. Martn and E. Verdaguer, Phys. Rev. D 60, 084008 (1999).

- Use the conformal transformation for the NK for conformally invariant fields to obtain the NK for the **Bunch-Davies** state in the co-moving de Sitter coordinates. (cosmology)

Use for cosmological problems

- Bunch-Davies vac are useful for stress-energy tensor fluctuations considerations in cosmological structure formation, e.g.,

  L. H. Ford, S. P. Miao, K.-W. Ng, R. P. Woodard, and C.-H. Wu, Phys. Rev. D 82, 043501 (2010).

- Gravity Waves from Quantum Stress Tensor Fluctuations in Inflation, e.g.,

  Chun-Hsien Wu, Jen-Tsung Hsiang, L. H. Ford, Kin-Wang Ng
  Phys. Rev. D 84, 103515 (2011)

NK for conformal fields for all conformally-flat spacetimes

We compute **exact expressions of the noise kernel** for conformally invariant scalar fields with respect to the conformal vacuum,

valid **for any arbitrary separation** (timelike, spacelike and null) of points for all conformally-flat spacetimes.

- Related recent work:

G. Perez-Nadal, A. Roura, and E. Verdaguer, JCAP 05 (2010) 036. minimal field in **de Sitter**,

H. T. Cho and B. L. Hu, Phys. Rev. D84, 044032 (2011)

massive fields in **N-dim AdS** space, arbitrary coupling.

- For the conformally invariant scalar field the noise kernel transforms as

$$\tilde{N}_{abc'd'}(x,x') = \Omega(x)^{-2} N_{abc'd'}(x,x') \Omega(x')^{-2}$$

- Thus, the noise kernel for the conformal vacuum of any conformally flat (Cartesian) metric is

$$N_{abc'd'}(x,x') = \Omega(x)^{-2}\Omega(x')^{-2}\left[\frac{\sigma_a\sigma_b\sigma_{c'}\sigma_{d'}}{48\pi^4\sigma^6} - \frac{\sigma_{(a}\sigma_{b)(c'}\sigma_{d')}}{24\pi^4\sigma^5} + \frac{4\sigma_{a(c'}\sigma_{d')b} - \eta_{ab}\eta_{c'd'}}{192\pi^4\sigma^4}\right]$$

$$\sigma = \frac{1}{2}\left(-\Delta t^2 + \Delta\vec{x}^2\right)$$

Static de Sitter

- de Sitter is a vacuum solution to the Einstein equations when a cosmological constant is included.

$$ds^2 = -dt^2 + e^{t/\alpha}[dx^2 + dy^2 + dz^2]$$

- However, written in the static coordinate system, the de Sitter metric is:

$$ds^2 = -\left(1 - \frac{\rho^2}{\alpha^2}\right) dT^2 + \frac{d\rho^2}{1 - \frac{\rho^2}{\alpha^2}} + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2$$

- This metric has a form similar to Schwarzschild, and exhibits a cosmological horizon at $\rho = \alpha$.
- This gives us an opportunity to test the Schwarzschild results.

- The exact noise kernel for the conformal vacuum in the static de Sitter coordinates:

$$
\begin{aligned}
N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x') \;=\; & \frac{1}{12\pi^4\left[\alpha^2\left(\sqrt{BB'}\,\tau-2\right)+2\rho\rho'\cos(\gamma)\right]^6} \\
& \times\Big\{\alpha^4\Big[-12\sqrt{BB'}\,\tau+BB'\left(\tau^2+14\right) \\
& -\left(2B+2B'-6\right)\left(\tau^2-1\right)\Big] \\
& +4\alpha^2\rho\rho'\cos(\gamma)\left(3\sqrt{BB'}\,\tau-2\left(\tau^2-1\right)\right) \\
& +2\rho^2\rho'^{\,2}\left(\tau^2-1\right)\cos(2\gamma)\Big\}
\end{aligned}
$$

$$
B \;=\; 1-\frac{\rho^2}{\alpha^2} \qquad\qquad \tau\equiv 2\cosh(\Delta T/\alpha)
$$

$$
\cos\gamma \;\equiv\; \cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')
$$

Part III Cosmology

Primordial cosmological perturbations problem via Stochastic Gravity:

- *Gives result equivalent at* linear order *to usual method of quantizing metric and inflaton perturbations*

- *But can treat quadratic orde perturbations which is needed in R2 trace anomaly driven (Starobinsky 1981) inflation*

A. Roura and E. Verdaguer, Phys. Rev. D 78, 064010 (2008).

SEMICLASSICAL EINSTEIN EQUATION

Renormalization introduces quadratic tensors

$$G_{ab}[g]+\Lambda g_{ab}-\alpha A_{ab}[g]-\beta B_{ab}[g]=\kappa\langle\hat{T}_{ab}[g]\rangle_{ren}$$

where

$$A^{ab}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}\int d^4x\sqrt{-g}\,C_{cdef}C^{cdef}$$

$$B^{ab}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}\int d^4x\sqrt{-g}\,R^2$$

$$T^{ab}=\nabla^a\phi\nabla^b\phi-\frac{1}{2}g^{ab}(\nabla^c\phi\nabla_c\phi+m^2\phi^2)+\xi(g^{ab}\nabla^c\nabla_c-\nabla^a\nabla^b+G^{ab})\phi^2$$

INFLUENCE FUNCTIONAL

- Open quantum system *(Feynman-Vernon 63)*

system h_{ab} ⟺ ϕ_j **environment**

$$F_{IF} \equiv e^{iS_{IF}} = \int D[\phi_+]D[\phi_-]\exp(S_m[\phi_+, g^+] - S_m[\phi_-, g^-])$$

$$S_{IF}(g+h^{\pm}) = \frac{1}{2}\int \langle \hat{T}_x \rangle [h_x] - \iint [h_x] H_{xy} \{h_y\} + \frac{i}{8}\iint [h_x] N_{xy} [h_y]$$

$$[h] \equiv h^+ - h^- \qquad \{h\} \equiv (h^+ + h^-)/2$$

(x,y denotes ab.cd)

$$H'_{xy} = \frac{1}{4}\mathrm{Im}\langle T^*(\hat{T}_x \hat{T}_y)\rangle - \frac{i}{8}\langle [\hat{T}_x, T_y]\rangle$$

$$\langle T^{(1)}_{ab}[g+h]\rangle_{ren} = -2\int d^4y \sqrt{-g}\, H_{abcd}(x,y) h^{cd}(y)$$

INFLUENCE FUNCTIONAL

- Closed Time Path effective action at tree level in metric pert.

$$\Gamma^{(0)}_{CTP}\left[h^+,h^-\right]=S_g\left[h^+\right]-S_g\left[h^-\right]+S_{IF}\left[h^+,h^-\right]+O\left(h^3\right)$$

S_g is EH action plus quadratic terms.

- Integral identity (Feynman Vernon 1963):

$$\mathrm{e}^{-\mathrm{Im}\,S_{IF}}\equiv\exp\left(-\frac{1}{8}\iint[h_x]N_{xy}\left[h_y\right]\right)\propto\int D\xi\exp\left(-\frac{1}{2}\iint\xi_x N^{-1}_{xy}\xi_y+\frac{i}{2}\int\xi_z[h_z]\right)$$

- Probability distribution functional of a classical stochastic field $\xi_{ab}(x)$

$$P[\xi]\propto e^{-\frac{1}{2}\iint\xi N^{-1}\xi}$$

$$e^{iS_{IF}[h^+,h^-]}=\int D\xi P[\xi]e^{i\left(\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi[h]\right)}\equiv\left\langle e^{i\left(\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi[h]\right)}\right\rangle_s$$

STOCHASTIC EFFECTIVE ACTION

- Define a stochastic effective action:

$$\Gamma_{stc}\left[h^{+},h^{-};\xi\right]=S_{g}\left[h^{+}\right]-S_{g}\left[h^{-}\right]+\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi_{z}\left[h_{z}\right]$$

- field equation from:

$$\left.\frac{\delta\Gamma_{stc}}{\delta h^{+}}\right|_{h^{\pm}=h}=0$$

⇒ the Einstein-Langevin equation

$$G_{ab}^{(1)}[g+h]=\kappa\langle\hat{T}_{ab}^{(1)}[g+h]\rangle_{ren}+\kappa\xi_{ab}[g]$$

SOLUTIONS OF EINSTEIN-LANGEVIN EQUATIONS

- These stochastic equations determine the correlations

$$h_{ab}(x) = h^0_{ab}(x) + \kappa \int d^4x' \sqrt{-g}\, G^{ret}_{abcd}(x,x')\, \xi^{cd}(x')$$

$$\langle h_{ab}(x) h_{cd}(y) \rangle_s = \langle h^0_{ab}(x) h^0_{cd}(y) \rangle_s + \kappa^2 \iint G^{ret}_{abef}(x,x') N^{efgh}(x',y') G^{ret}_{ghcd}(y',y)$$

Intrinsic fluctuations + **Induced fluctuations**

(flucts in the initial state)

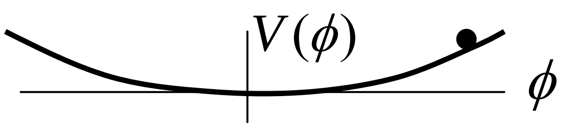
(due to matter field flucts)

- Stochastic metric correlations is equivalent to quantum metric correlations in 1/N: (Calzetta, Roura, Verdaguer)

$$\frac{1}{2} \langle \left\{ \hat{h}_{ab}(x), \hat{h}_{cd}(y) \right\} \rangle = \langle h_{ab}(x) h_{cd}(y) \rangle_s$$

STOCHASTIC GRAVITY AND PRIMORDIAL COSMOLOGICAL PERTURBATIONS

- Quantum fluctuations of inflaton are seeds for structure formation
- Simplest chaotic inflationary model (Linde): Massive minimally coupled inflaton field, initially at average value larger than Planck scale



$$\dot{\phi}^2 \ll V(\phi) \qquad m_P \lesssim \phi_0$$

$$L(\phi) = \frac{1}{2}\partial_a\phi\partial^a\phi + \frac{1}{2}m^2\phi^2$$

- Background inflaton field and FRW metric

$$\phi(\eta) = \langle\hat{\phi}\rangle \qquad ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$$

PERTURBATIONS

- Inflaton and scalar metric perturbations

$$\hat{\phi}(x) = \phi(\eta) + \hat{\varphi}(x) \qquad \langle \hat{\varphi} \rangle_g = 0$$

$$ds^2 = a^2(\eta)[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j]$$

- Einstein-Langevin equations

$$G^{(0)}_{ab}[g] - \kappa\langle \hat{T}^{(0)}_{ab}[g]\rangle + G^{(1)}_{ab}[h] - \kappa\langle T^{(1)}_{ab}[h]\rangle = \kappa\xi_{ab}[g]$$

zeroth order metric g is assumed to be quasi- de Sitter.

$$\hat{T}_{ab} = \tilde{\nabla}_a \hat{\phi} \tilde{\nabla}_b \hat{\phi} - \frac{1}{2}\tilde{g}_{ab}(\tilde{\nabla}_c \hat{\phi} \tilde{\nabla}^c \hat{\phi} + m^2 \hat{\phi}^2) \qquad \tilde{g} = g + h$$

$$\langle \xi(x)\xi(y)\rangle_s = \frac{1}{2}\langle \{\hat{t}(x), \hat{t}(y)\}\rangle[g] \qquad \hat{t} \equiv \hat{T} - \langle \hat{T} \rangle$$

STRESS TENSOR CORRELATIONS

$$\langle \text{涔}T[g+h]\rangle = \langle T[g+h]\rangle_{\phi\phi} + \langle \text{涔}T[g+h]\rangle_{\phi\varphi} + \langle T[g+h]\rangle_{\varphi\varphi}$$

$$\langle\{\text{涔}t, t\}\rangle[g] = \langle\{\text{涔}t, t\}\rangle[g]_{\phi^2\varphi^2} + \langle\{\text{涔}t, t\}\rangle[g]_{\varphi^2\varphi^2} \equiv \langle \xi^{(1)}\xi^{(1)}\rangle_s + \langle \xi^{(2)}\xi^{(2)}\rangle_s$$

- assume Gaussian state: $\langle\hat{\varphi}\rangle = 0 \qquad \langle \text{涔}\varphi\varphi\varphi\text{□}\rangle = 0$
- two independent stochastic sources: $\xi^{(1)}, \xi^{(2)}$
  independently conserved

- Including only the first (linear) term: $\langle\text{贩□}\rangle_{\phi^2\varphi^2}$

  we will show that the stochastic gravity formulation gives equivalent results as the traditional quantized metric and scalar field perturbations

E-L eqn for linear perturbations

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}^0_0\rangle_\Phi+\xi^0_0\right) = 3\mathcal{H}(\mathcal{H}\Phi+\Psi')-\nabla^2\Psi,$$

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}^i_0\rangle_\Phi+\xi^i_0\right) = \partial_i(\Psi'+\mathcal{H}\Phi),$$

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}^j_i\rangle_\Phi+\xi^j_i\right) = \Big[\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+\mathcal{H}\Phi'+\Psi''+2\mathcal{H}\Psi'+\frac{1}{2}\nabla^2 D\Big]\delta^j_i-\frac{1}{2}\delta^{jk}\partial_k\partial_i D.$$

where $\mathcal{H}=a'(\eta)/a(\eta)$, $D=\Phi-\Psi$, $\nabla^2=\delta^{ij}\partial_i\partial_i$

- Since $\langle \hat{T}_{ij} \rangle = 0, (i \neq j)$ ➡ $\xi_{ij} = 0, (i \neq j)$

➡ metric perturbations $\Phi = \Psi$

- Fourier transf. of 0i-component: (neglecting non-local term):

$$2k_i (H\Phi_k + \Phi'_k) = \kappa \xi_{k(0i)} \qquad H \equiv \frac{a'(\eta)}{a(\eta)}$$

- Retarded propagator for Φ_k

$$G^k_{ret}(\eta, \eta') = \frac{\kappa}{2k_i} \left(\theta(\eta - \eta') \frac{a(\eta)}{a(\eta')} + f(\eta, \eta') \right)$$

With $\Psi = \Phi$ we get for the ii component of E-L eqn:

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{T}_i^i\rangle_\Phi + \xi_i^i\right) = \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + 3\mathcal{H}\Phi' + \Phi''.$$

Two unknowns 1. scalar metric perturbations $\Phi(x)$

2. $\langle\hat{\varphi}\rangle_\Phi$ the expectation value of the quantum operator for the inflaton perturbations on the spacetime with the perturbed metric, $\langle\hat{\varphi}[g+h]\rangle$

These three equations reduce to two because of the Bianchi Identity, which holds here since the averaged and stochastic sources in the EL eqn are separately conserved.

the one hand, the conservation of $\langle\delta\hat{T}_{ab}\rangle_\Phi$ is equivalent to the Klein-Gordon equation for the expectation value $\langle\hat{\varphi}\rangle_\Phi$, which is completely analogous to Eq. (36):

$$\langle\hat{\varphi}\rangle_\Phi'' + 2\mathcal{H}\langle\hat{\varphi}\rangle_\Phi' - \nabla^2\langle\hat{\varphi}\rangle_\Phi + m^2a^2\langle\hat{\varphi}\rangle_\Phi - 4\phi'\Phi' + 2m^2a^2\phi\Phi = 0. \qquad (41)$$

On the other hand, the conservation of the stochastic source is a consequence of the conservation of the noise kernel, which in turn relies on the fact that the quantum operator for the inflaton perturbations $\hat{\varphi}[g]$ satisfies the Klein-Gordon equation on the background spacetime, $\left(\nabla_a\nabla^a - m^2\right)\hat{\varphi}(x) = 0$.

Equivalence with Quantum approach:

Can show that EL eqn reduces to (Roura and Verdaguer 2007)

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' - \nabla^2\Phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi = 0,$$

Same as the conventional approach via quantized linear perturbations, e.g., Eq. (6.48) of

V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

Comments:

1. In Fourier space nonlocal terms in the integro-differential equation in the spatial sector simplify to products. Non-locality in time in this equation disappears due to an exact cancellation of the different contributions from $\langle \delta \hat{\mathcal{T}}_a^b \rangle_\Phi$
2. Seems like there is no dependence on the stochastic source.
   But the solutions to ELeqn should also satisfy the constraint eqn at the initial time in addition to the dynamical eqn. The initial conditions for $\Phi_k(\eta_0)$ and $\Phi'_k(\eta_0)$ have dependence on the stochastic source.

CORRELATIONS FOR METRIC PERTURBATIONS

- Solutions of E-L equation:

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s = (2\pi)^2 \delta(\vec{k}+\vec{k}') \iint G_{ret}^{k} \langle \xi_k \xi_{k'} \rangle_s G_{ret}^{k'}$$

$$\langle \xi_k \xi_{k'} \rangle_s \equiv \frac{1}{2} \langle \{ t_k, t_{k'} \} \rangle$$

$$\langle \{ t_{0i}^{k}(\eta_1), t_{0i}^{-k}(\eta_2) \} \rangle = k_i k_i \phi'(\eta_1)\phi'(\eta_2) \langle \{ \varphi_k(\eta_1), \varphi_{-k}(\eta_2) \} \rangle$$

$$\langle \{ \varphi_k(\eta_1), \varphi_{-k}(\eta_2) \} \rangle = G_k^{(1)}(\eta_1, \eta_2)$$

is the Hadamard function for free scalar field on de Sitter, in Euclidean vacuum

$$a(\eta) = -\frac{1}{H\eta}; -\infty < \eta < 0$$

METRIC PERTURBATION CORRELATIONS

Computing $G_k^{(1)}$ perturbatively in m/m_P

assuming slow roll $\dot{\phi}(t) \sim -m_P^2(m/m_P)$

taking (rather insensitive to initial conds.) $\eta_0 \to -\infty$

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s \simeq 8\pi^2 \left(\frac{m}{m_P}\right)^2 k^{-3}(2\pi)^3 \delta(\vec{k}+\vec{k}')\cos[k(\eta-\eta')]$$

- Harrison-Zel'dovich scale inv. spectrum large scales $k\eta \leq 1$
- Amplitude of CMB anisotropies ➡ $\frac{m}{m_P} \sim 10^{-6}$
- Agreement with linear perturbations approach (Mukhanov 92)
- Stochastic gravity can go beyond linear app. in inflaton flucts. (Weinberg 05) and deal with Starobinsky (tr anomaly) inflation

Summary: Main Features

1. Semiclassical gravity depends on e.v. of q. stress tensor

S.G fails when flucts. of quantum stress tensor are large

2. Stochastic gravity incorporates these fluctuations (at Gaussian level) through the noise kernel acting as source for the Einstein-Langevin equation

3. Stochastic two-point metric correlations agree with quantum two-point metric correlations to order 1/N in large N expansion

4. Noise kernel the centerpiece for the exploration of **metric fluctuations and backreaction problems** of quantum fields in curved spacetimes.

- For Cosmological Perturbation and Structure Formation:
  Agreement with linear perturbations approach (e.g.,Mukhanov 92)
  But can go beyond linear order in inflaton fluctuations
  necessary for trace-anomaly driven inflations (e.g., Starobinsky 1980)

- Three recent papers from our group (with Paul Anderson, Jason Bates, HT Cho, A. Eftekharzadeh and Albert Roura) on the **calculation of NK, both exact expressions and under specific approximation**.

- In details, the case of **conformal fields in conformally flat spacetime,** in de Sitter, both wrt **Bunch-Davies vacuum and Gibbons-Hawking vacuum.**

- Relevance to work on structure formation and primordial gravitational waves in inflationary cosmology and fluctuations of event horizon and backreaction of Hawking radiation in quasi-static black holes.

Thank You!   谢谢！

Hope the finest Russian / Chinese
Theoretical physics tradition will continue.

An advice for the younger generation:
-Try to r*ide over the adverse effects of cheap salesman **Aca-Business** mentality from the West, and **Rotten Politics** of the East.*

Quantum Open System

Closed System: **Density Matrix** $\hat{\rho}(t) = \mathcal{J}(t, t_i)\hat{\rho}(t_i).$

$\mathcal{J}(x, \mathbf{q}, x', \mathbf{q}', t \mid x_i, \mathbf{q}_i, x'_i, \mathbf{q}'_i, t_i)$ is the (**unitary) evolutionary operator** of the system from initial time t_I to time t.

OPEN SYSTEM: System (s) interacting with an Environment (e) or Bath (b): Integrate out (coarse-graining) the bath dof renders the system open. Its evolution is described by the Reduced Density Matrix

$$\rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, \mathbf{q}; x', \mathbf{q}')\delta(\mathbf{q} - \mathbf{q}')$$

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i\ \mathcal{J}_r(x, x', t \mid x_i, x'_i, t_i)\ \rho_r(x_i, x'_i, t_i\ ).$$

Influence Functional

Assume **factorizable condition** between the system (s) and the bath (b) initially

$$\hat{\rho}(t = t_i) = \hat{\rho}_s(t_i) \times \hat{\rho}_b(t_i),$$

:

Evolutionary operator for the reduced density matrix is

$$\mathcal{J}_r(x_f, x'_f, t \mid x_i, x'_i, t_i) = \int_{x_i}^{x_f} Dx \int_{x'_i}^{x'_f} Dx' \, \exp\left(\frac{i}{\hbar}\left\{S[x] - S[x']\right\}\right) \, \mathcal{F}[x, x']$$

Influence Functional

$$\mathcal{F}[x, x'] = \int_{-\infty}^{+\infty} d\mathbf{q}_f \int_{-\infty}^{+\infty} d\mathbf{q}_i \int_{-\infty}^{+\infty} d\mathbf{q}'_i \int_{\mathbf{q}_i}^{\mathbf{q}_f} D\mathbf{q} \int_{\mathbf{q}'_i}^{\mathbf{q}_f} D\mathbf{q}'$$

$$\exp\left(\frac{i}{\hbar}\left\{S_b[\mathbf{q}] + S_{\text{int}}[x, \mathbf{q}] - S_b[\mathbf{q}'] - S_{\text{int}}[x', \mathbf{q}']\right\}\right) \times \rho_b(\mathbf{q}_i, \mathbf{q}'_i, t_i)$$

Influence Action

$$= \exp\left(\frac{i}{\hbar}\delta\mathcal{A}[x, x']\right)$$

Quantum Brownian Motion II

System (S): quantum oscillator *with time dependent natural frequency*
Environment (E) : n-quantum oscillators
with time-dependent natural frequencies = Scalar Field
Coupling: c_n F (x) q_n.

$$S[x,\mathbf{q}] = S[x] + S_E[\mathbf{q}] + S_{\rm int}[x,\mathbf{q}]$$
$$= \int_0^t ds \left[\frac{1}{2} M(s)[\dot{x}^2 + B(s)x\dot{x} - \Omega^2(s)x^2] \right.$$
$$\left. + \sum_n \left\{ \frac{1}{2} m_n(s)[\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2] + \sum_n \left(-c_n(s)F(x)q_n \right) \right. \right]$$

Influence functional for a Paramp

$$\mathcal{F}[x,x'] = \exp\Big\{-\frac{i}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\mu(s,s')\Big[F(x(s')) + F(x'(s'))\Big]$$
$$-\frac{1}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\nu(s,s')\Big[F(x(s')) - F(x'(s'))\Big]\Big\}$$

$$\Sigma(s) = \frac{1}{2}\Big(F(x(s)) + F(x'(s))\Big),$$

Dissipation μ and Noise ν Kernels

$$\Delta(s) = F(x(s)) - F(x'(s)),$$

Δ (s) μ (s,s') Σ (s')

$$\mathcal{F}[x,x'] = \exp\Big\{\frac{i}{\hbar}\int_{t_i}^{t} ds\Delta(s)\langle\xi(s)\rangle \quad -\frac{1}{\hbar^2}\int_{t_i}^{t} ds \int_{t_i}^{s} ds'\Delta(s)\Delta(s')C_2(s,s')\Big\}$$

$$\langle\bar{\xi}(t)\bar{\xi}(t')\rangle \;=\; C_2(s,s') \equiv \hbar\nu(s,s')$$

Langevin Equation:::

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - 2\frac{\partial F(x)}{\partial x}\int_{t_i}^{t}\mu(t,s)F(x(s))ds = -\frac{\partial F(x)}{\partial x}\bar{\xi}(t)$$

Noise and Dissipation Kernels

Equation of Motion for the amplitude function of a Parametric Oscillator

$b_n = 0$ and $m = 1$ $\kappa_n = m_n(t_i)\omega_n(t_i)$ $\ddot{X}_n + \omega_n^2(t)X_n = 0,$

$$\mu(s,s') = \frac{i}{2}\int_0^\infty d\omega I(\omega,s,s')\Big[X_\omega^*(s)X_\omega(s') - X_\omega(s)X_\omega^*(s')\Big],$$

$$\nu(s,s') = \frac{1}{2}\int_0^\infty d\omega I(\omega,s,s')\coth\left(\frac{\hbar\omega(t_i)}{2k_BT}\right)\Bigg[\cosh 2r(\omega)\Big[X_\omega^*(s)X_\omega(s') + X_\omega(s)X_\omega^*(s')\Big]$$

$$-\sinh 2r(\omega)\Big[e^{-2i\phi(\omega)}X_\omega^*(s)X_\omega^*(s') + e^{2i\phi(\omega)}X_\omega(s)X_\omega(s')\Big]\Bigg].$$

$$I(\omega,s,s') = \sum_n \delta(\omega-\omega_n)\frac{c_n(s)c_n(s')}{2\kappa_n}$$

Spectral Density Function

$I(\omega) \sim \omega^n$ n=1: Ohmic, n>1 Supra Ohmic; n<1 Subohmic

Squeezed and Rotation parameters: $\hat{\rho}_b(t_i) = \prod_n \hat{S}_n(r(n),\phi(n))\hat{\rho}_{th}\hat{S}_n^\dagger(r(n),\phi(n))$

e.g., for an initial squeezed thermal bath

Stochastic Equations

Master Equation:
(Non- Markovian)

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}_r(t) = [\hat{H}_{\text{ren}}, \hat{\rho}] + iD_{pp}[\hat{x}, [\hat{x}, \hat{\rho}]] + iD_{xx}[\hat{p}, [\hat{p}, \hat{\rho}]]$$

$$+ iD_{xp}[\hat{x}, [\hat{p}, \hat{\rho}]] + iD_{px}[\hat{p}, [\hat{x}, \hat{\rho}]] + \Gamma[\hat{x}, \{\hat{p}, \hat{\rho}\}],$$

$$\hat{H}_{\text{ren}} = \frac{\hat{p}^2}{2M(t)} - \frac{B(t)}{4}(\hat{p}\hat{x} + \hat{x}\hat{p}) + \frac{M(t)}{2}\Omega_{\text{ren}}(t)\hat{x}^2.$$

Wigner Function:

$$F_W(\Sigma, p, t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip\Delta/\hbar}\left\langle \Sigma - \frac{\Delta}{2}\middle|\hat{\rho}\middle|\Sigma + \frac{\Delta}{2}\right\rangle d\Delta,$$

Fokker-Planck or Wigner Equation: (Non-Markovian)

$$\frac{\partial}{\partial t}F_W(\Sigma, p, t) = \left[-\frac{p}{M(t)}\frac{\partial}{\partial \Sigma} + \frac{1}{2}M(t)\Omega^2_{\text{ren}}(t)\;\Sigma\frac{\partial}{\partial p} + \Gamma(t)\frac{\partial}{\partial p}p - 2D_{pp}(t)\frac{\partial^2}{\partial p^2}\right.$$

$$\left. -\hbar D_{xx}(t)\frac{\partial^2}{\partial \Sigma^2} + 2\Big(D_{xp}(t) + D_{px}(t)\Big)\frac{\partial^2}{\partial \Sigma \partial p}\right]F_W(\Sigma, p, t).$$

Use of exact expressions of the noise kernel for fluc - bkrn problems in early universe & black holes

1. Examine its behavior in the region **near the cosmological horizon**. (they are all finite)

   From the well-known mapping we can see the

2. behavior **of NK in the Schwarzschild spacetime** wrt the Hartle-Hawking vacuum near the horizon.

3. Check the **validity range of quasi-local expansion** in evaluating the NK as done in

   [A. Eftekharzadeh, J. D. Bates, A. Roura, P. R. Anderson, and B. L. Hu, Phys. Rev. D85, 044037 (2012).