# Free fermion and wall-crossing

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# Motivation

- For string theorists
  It is called BPS states counting which means counting the D brane bound states.

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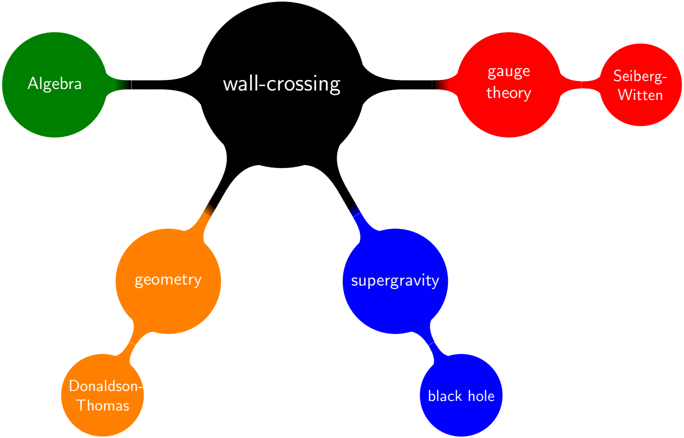
- For string theorists
  It is called BPS states counting which means counting the D brane bound states.
- For mathematician
  Invariants of moduli spaces of virtual dimension zero associate with Calabi-Yau 3-fold $X$
  1. holomorphic curves in $X$ with fixed genus and degree, e.g. Gromov-Witten invariants
  2. coherent sheaves with a fixed Chern character, e.g. Donaldson-Thomas (DT) invariants

# The world of wall



# Kontsevich-Soibelman wall-crossing formula

There is a symplectomorphism transformation for functions over algebraic torus over lattice $\Gamma$

$$U_\gamma : X_{\gamma'} \longrightarrow X_{\gamma'}(1+\sigma(\gamma)X_\gamma)^{\langle\gamma',\gamma\rangle} \tag{1}$$

where $U_\gamma$ can be expressed as the operator

$$U_\gamma = \exp\left(\sum_n \frac{\sigma(n\gamma)}{n^2}\{X_{n\gamma},\cdot\}\right) \tag{2}$$

Wall-crossing formula

$$\prod_\gamma^{\curvearrowright} U_\gamma^{\Omega(\gamma,u_+)} = \prod_\gamma^{\curvearrowright} U_\gamma^{\Omega(\gamma,u_-)} \tag{3}$$

# Motivic wall-crossing formula

On the quantum algebraic torus the associative algebra is generated by $\hat{e}_\gamma$ s.t.

$$[\hat{e}_{\gamma_1}, \hat{e}_{\gamma_2}] = \left( q^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} - q^{-\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} \right) \hat{e}_{\gamma_1+\gamma_2} \tag{4}$$

The wall-crossing formula is

$$\prod_{\gamma}^{\curvearrowright} A_\gamma^{mot}(u_+) = \prod_{\gamma}^{\curvearrowright} A_\gamma^{mot}(u_-) \tag{5}$$

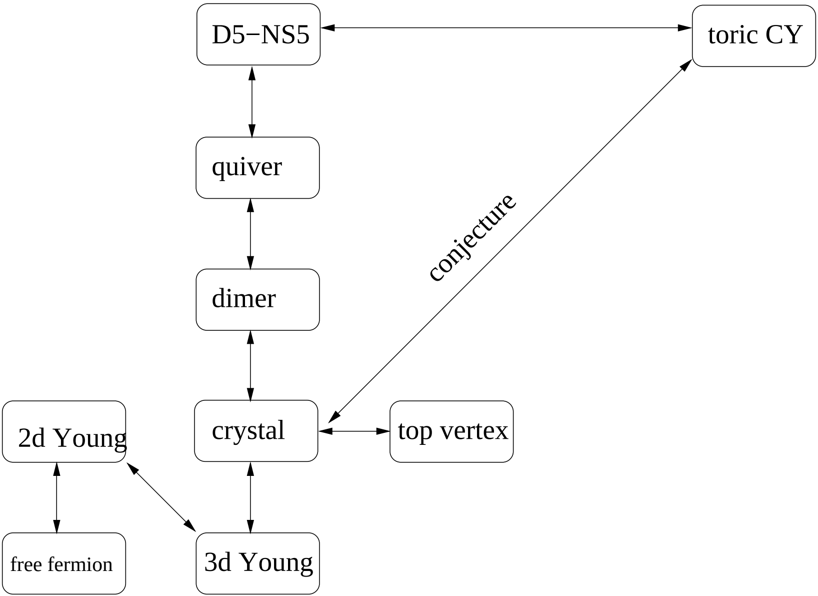
where $A_\gamma^{mot}$ is a quantum analog of the classical symplectomorphism $U_\gamma^{\Omega(\gamma)}$.

Dualities of BPS counting

Free fermion for $\mathbb{C}^3$

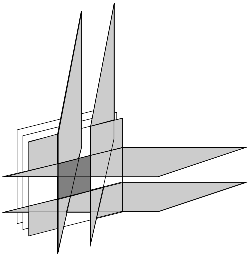
Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

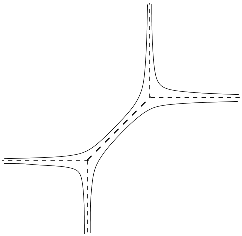
Open questions


D5−NS5
toric CY
quiver
conjecture
dimer
2d Young
crystal
top vertex
free fermion
3d Young

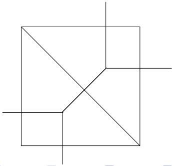
# D5-NS5 brane and toric Calabi-Yau geometry

T-duality (mirror symmetry) between D5-NS5 brane configuration and toric diagram

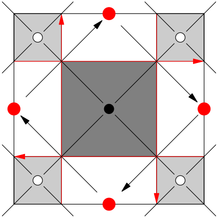
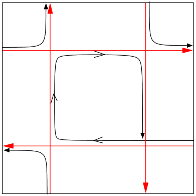
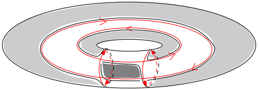




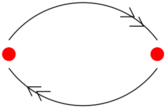
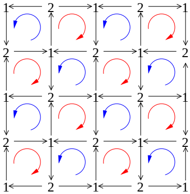
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D5 | * | * | * | * | | – | | – | | |
| NS5 | * | * | * | * | – | – | | | | |
| NS5 | * | * | * | * | | | – | – | | |

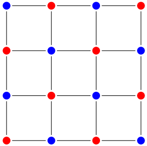


# Quiver and dimer model

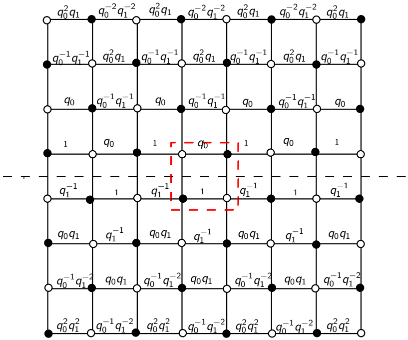
The quiver, universal quiver, and dimer diagram for conifold are



# Crystal

(e) a screenshot of paper by Young and Bryan arXiv:0802.3948




(f) dimer assigned with weight

Conjecture:
The partition function of statiscal model of crystal melting is related to the BPS partition function.

Dualities of BPS counting

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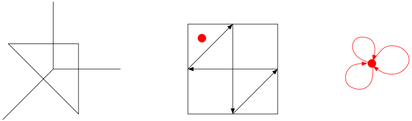
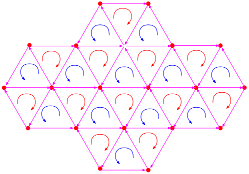
Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

Open questions

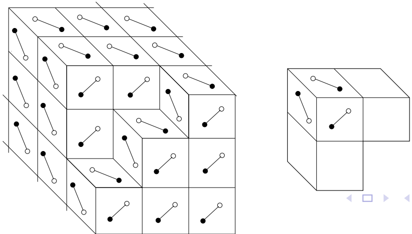
## Build up the quiver diagram for $\mathbb{C}^3$



## The universal quiver for $\mathbb{C}^3$

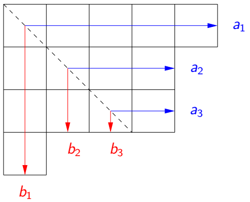


## The dimer model for $\mathbb{C}^3$



# Free fermion and 2D Young diagram correspondence

- 2D Young diagram $\lambda$




- The correspondence

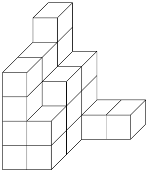
$$\lambda \Leftrightarrow |\lambda\rangle = \prod_{i=1}^{d(\lambda)} \psi_{-(a_i)}\psi^*_{-(b_i)}|0\rangle \tag{6}$$

where

$$a_i = \lambda_i - i + \frac{1}{2}, \quad b_i = \lambda^t_i - i + \frac{1}{2} \tag{7}$$

# 3D Young diagram $Y_3$

- Plane partitions
  Definition: 2D Young diagram with weakly decreasing number filling in rows and columns



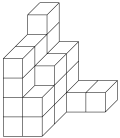
| | | | |
|---|---|---|---|
| 5 | 3 | 1 | 1 |
| 4 | 3 | | |
| 4 | 2 | | |

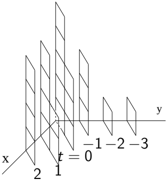
- MacMahon function is the generating function of $Y_3$

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \sum_{\pi \in \text{all } Y_3} q^{|\pi|} \tag{8}$$

# 2D and 3D Young diagram

- The diagonal diagram






- Interlacing condition of the diagonal slices

$$\begin{cases} \lambda(t) \succ \lambda(t+1) & t > 0 \\ \lambda(t) \succ \lambda(t-1) & t < 0 \end{cases} \tag{9}$$

where

$$\lambda \succ \mu \quad \Longleftrightarrow \quad \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots \tag{10}$$

- Vertex operators

$$\Gamma_{\pm}(x) = \exp\left\{\sum_{n=1}^{\infty} \frac{x^n}{n}\alpha_{\pm n}\right\} \tag{11}$$

$\alpha_{\pm n}$ are the creation and annihilation operator for the bosonization of free fermion.

- $\Gamma_-$ can be treated as the creation operator while $\Gamma_+$ the annihilation one of nearest neighbor slices, i.e.

$$\Gamma_{\substack{-\\+}}(1)|\mu\rangle = \sum_{\substack{\lambda \succ \mu \\ \lambda \prec \mu}} |\lambda\rangle, \tag{12}$$

Commutation relation

$$\Gamma_+(x)\Gamma_-(y) = \frac{1}{1-xy}\Gamma_-(y)\Gamma_+(x) \tag{13}$$

# Crystal melting for $\mathbb{C}^3$

- The statistics of a melting cubic crystal is identified with the partition function of D0-D6 bound states of $\mathbb{C}^3$
- MacMahon partition function

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \qquad (14)$$
$$\langle \underbrace{q^{L_0}\Gamma_+(1)\cdots q^{L_0}\Gamma_+(1)}_{\infty} q^{L_0} \underbrace{\Gamma_-(1)q^{L_0}\cdots\Gamma_-(1)q^{L_0}}_{\infty} \rangle$$

where we perform commutation relation of vertex operators.

$$\prec\lambda(-2)\prec\lambda(-1)\prec\lambda(0)\succ\lambda(1)\succ\lambda(2)\succ$$

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Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

Open questions

# Marginal stability wall

- SUSY condition for mass and central charge of a state

$$M \geq |Z(\gamma)| \tag{15}$$

  when "$=$", it is called a BPS state.
- Marginal stability wall

$$\gamma = \gamma_1 + \gamma_2, \quad Z(\gamma) = Z(\gamma_1) + Z(\gamma_2) \tag{16}$$

  2-particle BPS states $M_{2-\mathrm{particle}} = |Z_1 + Z_2|$ satisfies

$$M_{2-particle} = |Z_1 + Z_2| \leq |Z_1| + |Z_2| = M_1 + M_2 \tag{17}$$

  Therefore 2-particle BPS states are separated from 1-particle BPS states $M_1 = |Z_1|, M_2 = |Z_2|$ unless $|Z_1 + Z_2| = |Z_1| + |Z_2|$, i.e., at the wall.

# D0-D2-D6 bound states

- Charge lattice

| D0 | D2 | D4 | D6 |
|---|---|---|---|
| $H_0$ | $H_2$ | $H_4$ | $H_6$ |
| $q_0$ | $Q$ | $P$ | $p^0$ |
| $n$ | $-m$ | $0$ | $1$ |

$\gamma = ndV - m\beta + 1$

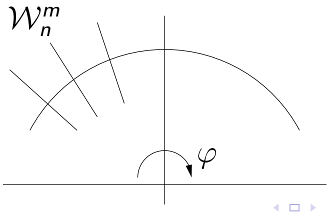
- Central charge of BPS states

$$Z(\gamma) = -\int_X \gamma \wedge e^{-t} \qquad (18)$$

where the complexified Kähler moduli is $t = z\mathcal{P} + \Lambda e^{i\varphi}\mathcal{P}'$

- Marginal stability walls in the moduli space of BPS states

$$ArgZ(\gamma) = ArgZ(\gamma_1) \qquad (19)$$



# Index of BPS states

Define an index by

$$\Omega(\gamma) = \mathrm{Tr}_{\mathcal{H}^{\gamma}_{BPS,u}}(-1)^{2J}, \tag{20}$$

where $\gamma \in \Gamma$ the charge lattice and $J \in \mathfrak{so}(3)$ is the generator of rotations around any axis.

$$\begin{aligned}\Omega(\gamma) &\sim \text{Euler characteristic of the stable sheaf moduli space}\\ &\sim \text{Donaldson-Thomas Invariants}\end{aligned}$$

Refined index is defined by [Dimofte & Gukov arXiv:0904.1420]

$$\Omega^{\mathrm{ref}}(\gamma; y) = \mathrm{Tr}_{\mathcal{H}^{\gamma}_{BPS,u}}(-y)^{2J} \tag{21}$$

$$\Omega^{\mathrm{ref}}(\gamma; y \to 1) = \Omega(\gamma) \tag{22}$$

# Some description of "refined"

| refined | | quantum | | motivic |
|---|---|---|---|---|
| $y$ | $\leftrightarrow$ | $-q^{\frac{1}{2}}$ | $\leftrightarrow$ | $\mathbb{L}^{\frac{1}{2}}$ |

- Physically we distinguish left and right spin, (corresponding to turing on $\Omega$ background in Nekrasov's formalizm)

- Mathematically

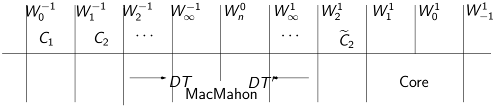
  $\Omega^{\mathrm{ref}}(\gamma; y)$

  $\sim$ Poincare polynomial of the stable sheaf moduli space

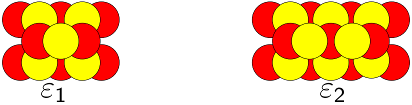
  $\sim$ Refined/Motivic Donaldson-Thomas Invariants

# Wall-crossing for the resolved conifold

- Wall




- 2-colored stone diagram for chamber 0 and 1




- The relation between 2D partition and free fermion is preserved. But for different slices the vertex operators have different arguments.

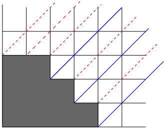
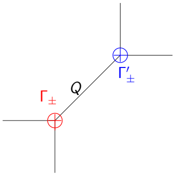


Figure: The red (dotted) lines denote the left or right moving of the slices, and the blue (solid) lines denote the up or down moving

# Crystal melting for resolved conifold

Vertex operators

Toric diagram




$$\Gamma_\pm(x) = \exp\left\{\sum_n \frac{x^n}{n}\alpha_{\pm n}\right\}$$

$$\Gamma'_\pm(x) = \exp\left\{\sum_n \frac{(-1)^{n-1}x^n}{n}\alpha_{\pm n}\right\}$$

$$\Gamma'_{\substack{-\\+}}(1)|\mu\rangle = \sum_{\substack{\lambda^t \succ \mu^t \\ \lambda^t \prec \mu^t}} |\lambda\rangle,$$

# How do we count refined in the crystal melting?

arXiv: 1010.0348 collaborated with Haitao Liu

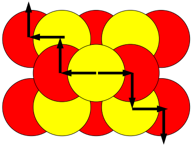
- Arrow diagrams for chamber $n$ are

$$\Gamma_+\left[q_1^{i-1}(-Q)^{\frac{1}{2}}\right] \quad \Gamma'_+\left[q_1^{i-\frac{1}{2}+n}q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \quad \Gamma_-\left[q_2^{j+n}(-Q)^{-\frac{1}{2}}\right] \quad \Gamma'_-\left[q_1^{\frac{1}{2}}q_2^{j-\frac{1}{2}}(-Q)^{\frac{1}{2}}\right]$$

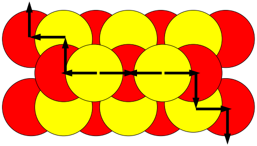
$$\Uparrow \qquad\qquad \Leftarrow \qquad\qquad \Rightarrow \qquad\qquad \Downarrow$$

Figure: Arrow diagrams for chamber $n$ of the conifold

- Stone diagrams with arrows are



n=0



n=1

# Vertex Operators

If we define $\overline{A}_{\pm}(x)$ by

$$\overline{A}_{+}(x) := \widehat{Q}_{01}^{\frac{1}{2}}\widehat{Q}_{1}^{\frac{1}{2}}(\widehat{Q}_{01}^{\frac{1}{2}}\Gamma_{+}(x)\widehat{Q}_{01}^{-\frac{1}{2}})\widehat{Q}_{1}(\widehat{Q}_{02}^{\frac{1}{2}}\Gamma'_{+}(x)\widehat{Q}_{02}^{-\frac{1}{2}})\widehat{Q}_{1}^{-\frac{1}{2}}\widehat{Q}_{01}^{\frac{1}{2}},$$

$$\overline{A}_{-}(x) := \widehat{Q}_{02}^{\frac{1}{2}}\widehat{Q}_{1}^{\frac{1}{2}}(\widehat{Q}_{02}^{\frac{1}{2}}\Gamma_{-}(x)\widehat{Q}_{02}^{-\frac{1}{2}})\widehat{Q}_{1}(\widehat{Q}_{01}^{\frac{1}{2}}\Gamma'_{-}(x)\widehat{Q}_{01}^{-\frac{1}{2}})\widehat{Q}_{1}^{-\frac{1}{2}}\widehat{Q}_{02}^{\frac{1}{2}}$$

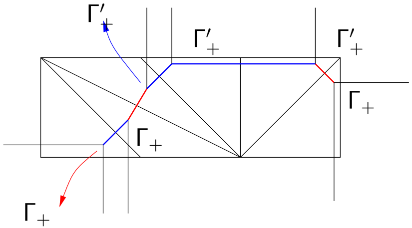
Then we can show that

$$\begin{aligned} Z_{NCDT}^{cystal} &:= \langle 0|\overline{A}_{+}(1)\cdots\overline{A}_{+}(1)\overline{A}_{-}(1)\cdots\overline{A}_{-}(1)|0\rangle \\ &= Z_{BPS}^{ref}(q_1,q_2,Q)|_{NCDT} \text{ (up to refined MacMahon).} \end{aligned}$$

For chamber $n$ we construct

$$\begin{aligned}
\mathcal{Z}_{crystal} &= \langle 0| \prod_{i=1}^{\infty} \Gamma_{+}\left[q_1^{i-1}(-Q)^{\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{i-\frac{1}{2}+n} q_2^{\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\times \Gamma_{-}\left[q_2(-Q)^{-\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{n-\frac{1}{2}} q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\times \Gamma_{-}\left[q_2^2(-Q)^{-\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{n-\frac{3}{2}} q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\cdots \times \Gamma_{-}\left[q_2^n(-Q)^{-\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\times \prod_{j=1}^{\infty} \Gamma_{-}\left[q_2^{j+n}(-Q)^{-\frac{1}{2}}\right] \Gamma'_{-}\left[q_1^{\frac{1}{2}} q_2^{j-\frac{1}{2}}(-Q)^{\frac{1}{2}}\right] |0\rangle \\
&= M_{\delta=1}(q_1, q_2) M_{\delta=-1}(q_1, q_2) \prod_{i,j=1}^{\infty} (1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q) \\
&\quad \prod_{\substack{i+j>n+1 \\ i,j\geq 1}}^{\infty} (1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q^{-1})
\end{aligned}$$

# Generalized conifold




We have some progress on constracting the free fermion formalism of crystal melting for NCDT chamber and DT chamber and computing the partition function of D0-D2-D6 bound states.

The refined MacMahon function $\mathcal{M}(q_1, q_2)$ is defined by [Behrend, Bryan, Szendrői arXiv:0909.5088]

$$\mathcal{M}(q_1, q_2) = \prod_{d=0}^{6} \left( M_{\frac{d-3}{2}}(q_1, q_2) \right)^{(-1)^d b_d},$$

where $b_d$ is the Betti number of the Calabi-Yau threefold $X$ of degree $d$ and $M_\delta(q_1, q_2)$ is the refined MacMahon function defined by

$$M_\delta(q_1, q_2) = \prod_{i,j=1}^{\infty} (1 - q_1^{i-\frac{1}{2}+\frac{\delta}{2}} q_2^{j-\frac{1}{2}-\frac{\delta}{2}})^{-1}.$$

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Open questions

- Refined MacMahon
  We have seen that by using crystal melting model we can recover the refined BPS partition function except for the refined MacMahon function (defined by Behrend et. al.). Is it possible to adjust the counting weight to recover the exact refined BPS partition function?

- Crystal and motivic DT
  What is the intrinsic reason that counting crystal can reproduce the refined BPS partition function?

- Invariants and symmetries
  How is Heisenberg algebra categorification going to enter this story? Namely, how can we find operators which satisfy KS motivic wall-crossing formula?

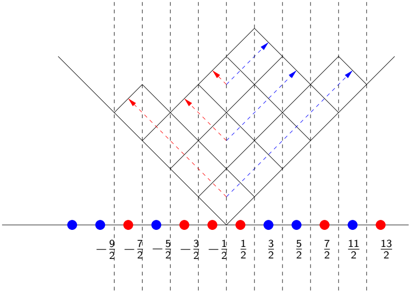
# Next: Topological string/gauge theory duality

- Geometric engineering
  arXiv: hep-th/9609239 Katz, Klemm, and Vafa
  Topological string theory on a Calabi-Yau with ADE type of singularities $\longleftrightarrow$ gauge theory of gauge group of the same ADE type

- Topological strings and Nekrasov's formulas
  arXiv: hep-th/0310235 Eguchi and Kanno

# Free fermion and gauge theory

arXiv: hep-th/0306238 Nekrasov and Okounkov
For $U(1)$ gauge field we can build the Maya diagram for a state $|\mu\rangle$ (charge is 0)



For $SU(N)$ group we need N-component fermion field

$$\psi_k^{(r)} = \psi_{N(k+\xi_r)}, \quad \psi_\ell^{(s)*} = \psi^*_{N(\ell-\xi_s)} \tag{23}$$

where $\xi_r = \frac{1}{N}(r - \frac{N+1}{2})$.

$$\{\psi_k^{(r)}, \psi_\ell^{(s)*}\} = \delta_{r,s}\delta_{k+\ell,0} \tag{24}$$

The sets of charged Maya diagrams of $\mu$ and $\lambda^{(r)}$ satisfy

$$\{x_i(\mu) + n; i \geq 1\} = \{N(x_{i_r}(\lambda^{(r)}) + p_r + \xi_r); i \geq 1\} \tag{25}$$

arXiv: hep-th/0412327 Maeda, Nakatsu, Takasaki and Tamakoshi

*Thank you!*