The black hole entropy in the critical gravity

[based on 1109.5486 Son, Eune, and Kim]

Wontae Kim

Center for Quantum Spacetime and Department of Physics
Sogang University, Seoul, Republic of Korea

28 Dec. 2011

Institute of Theoretical Physics, Chinese Academy of Science

Outline

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1. Introduction
2. (Classical) Euclidean action formulation
3. (Semiclassical) Euclidean action formulation
4. Entropy of a Schwarzschild-anti de Sitter black hole
5. Conclusion and discussions

Preliminary

- Since the general relativity is nonrenormalizable, there have been extensive studies for quantum theory of gravity such as string theory, conventional perturbative gravity, loop gravity and so on.
- A perturbatively renormalizable gravity theory can be built by adding quadratic curvature terms (R^2) to the Einstein gravity $\rightarrow$ Changing the mass dimensions of couplings. [Stelle '77 '78]
$\rightarrow$ POWER COUNTING RENORMALIZABLE
- However, theories including higher-order time-derivative terms should endure massive ghost modes.

$$\frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \cdots = \frac{1}{k^2} - \frac{1}{k^2 - G_N^{-1}}$$

(NB) In 2+1 dimensional topologically massive gravity with a cosmological constant, there exists a critical point such that the massive mode (ghost) becomes massless and carries no energy, so that the problem can be solved. (cf. tachyon-classical, ghost-quantum) [Li, Song & Strominger '08]

Chiral gravity

- Similarly, in 3+1 dimensional quadratic gravity with a cosmological constant, one can find a critical point, where the massive ghost mode disappears.
- The *critical gravity* can be defined by

$$I_{CG}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - 2\Lambda + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right],$$

where Λ is a cosmological constant.[Lu & Pope, PRL**106**, 181302 (2011)]
(NB) $\alpha + 3\beta = 0$ (decoupling condition for the scalar graviton)

- At the critical point, $\alpha = (3/2)\Lambda$, in spite of the renormalizability without the massive ghost, it becomes trivial in that the entropy of a Schwarzschild-anti de Sitter (SAdS) black hole which is a solution to this theory become zero.
(NB)This result can be confirmed by the Euclidean action formulation of the black hole thermodynamics. [Gibbons & Hawking '77 '78, Hawking & Page '83]

Motivation

- Bekenstein: the intrinsic entropy of a black hole which is proportional to the surface area at the event horizon [Bekenstein '72 '73 '74].
  Hawking: the quantum field theoretic calculation for the Schwarzschild black hole [Hawking '75]
- 't Hooft: the area law of black holes using the brick wall method ['t Hooft '85]
  (Motivation)
  Brick wall method $\rightarrow$ Area law
  Euclidean action formulation $\rightarrow 0$
- We would like to resolve this issue and study how to derive the entropy satisfying the area law in the Euclidean action formulation.

Strategy

- First task is to get a nontrivial free energy by taking into account higher-order loop corrections in the Euclidean path integral.
  (NB) The fluctuation of the metric field will be ignored for simplicity, i.e. our calculations will be performed in semiclassical approximations.
- So, we consider the one loop correction of the scalar degrees of freedom around the black hole and relate the Euclidean action formulation to the brick wall method semiclassically.
- Eventually, the (one-loop) free energy turns out to be nontrivial even at the critical condition. It is actually compatible with the free energy obtained from the brick wall method.

Outline

action

(1) The action: $I_{\rm tot} = I_{\rm CG} + I_\phi$, where

$$I_\phi[g,\phi] = -\int d^4x\sqrt{-g}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right].$$

For $\phi = 0$, the SAdS black hole is just a classical solution to this model.

(2) The line element: $ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Omega_2^2$ with

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2 = \left(1 - \frac{r_h}{r}\right)\left[1 - \frac{\Lambda}{3}(r^2 + r_h r + r_h^2)\right]$$

where $M = (r_h/2G)\left(1 - \Lambda r_h^2/3\right) > 0$ is the mass parameter of the black hole, $\Lambda < 0$ is the cosmological constant, and r_h is the radius of the horizon.

boundary terms

(3) The Euclidean action with an auxiliary field $f_{\mu\nu}$ [Hohm & Tonni '10]:

$$I_{\rm CG} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\left[R - 2\Lambda + f^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{4\alpha}f^{\mu\nu}\left(f_{\mu\nu} - g_{\mu\nu}f\right)\right],$$

$$I_{\rm B} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{\gamma}\left[2K + \hat{f}^{ij}\left(K_{ij} - \gamma_{ij}K\right)\right],$$

where γ_{ij} and K_{ij} are the induced metric and the extrinsic curvature of the boundary, respectively. And $\hat{f}^{ij}$ in the boundary term is defined as $\hat{f}^{ij} = f^{ij} + f^{ri}N^j + f^{rj}N^i + f^{rr}N^iN^j$ with $N^i = -g^{ri}/g^{rr}$ for the hypersurface described by $r = r_0$. In the Euclidean geometry, the Euclidean time is defined by $\tau = it$ and should be identified by $\tau = \tau + \beta_H$ to avoid a conical singularity at the event horizon, where β_H is the inverse of the Hawking temperature.

entropy

(4) The (classical) free energy:

$$F^{(0)} = \beta_H^{-1}(I - I_{\text{vacuum}}) = [1 - 2\alpha\Lambda/3]\,\frac{r_h}{4G}\left(1 + \frac{\Lambda}{3}r_h^2\right),$$

where $I_{\text{CGB}} = I_{\text{CG}} + I_{\text{B}}$ and $I_{\text{vacuum}} = I|_{M=0}$.

(5) The Hawking temperature:

$$T_H = \beta_H^{-1} = \frac{1 - \Lambda r_h^2}{4\pi r_h}.$$

(6) The entropy and the energy of the black hole:

$$S^{(0)} = \beta_H^2 \frac{\partial F^{(0)}}{\partial \beta_H} = [1 - 2\alpha\Lambda/3]\,\frac{\pi r_h^2}{G},$$

$$E^{(0)} = F^{(0)} + \beta_H^{-1} S^{(0)} = [1 - 2\alpha\Lambda/3]\,\frac{r_h}{2G}\left(1 - \frac{\Lambda}{3}r_h^2\right),$$

which are exactly same with those obtained in Ref. [Lu & Pope '11].

critical condition

- Note that the overall factor $[1 - 2\alpha\Lambda/3]$ vanishes at the critical point $\alpha = 3/2\Lambda$.
- Thus, we can confirm that the energy and the entropy of the SAdS black hole at the critical point vanish in the Euclidean action formulation.
  (NB) The entropy from the brick wall method satisfies the area law.
- In what follows, we shall show that the semiclassical treatment of the Euclidean action formulation can be related to the brick wall method.

Outline

Formal definition of the partition function

(7) The (semiclassical) partition function:

$$
\begin{aligned}
Z[g] &= Z^{(0)}[g] Z^{(1)}[g] \\
&= \exp\left(-\beta F^{(0)}\right) \exp\left(-\beta F^{(1)}\right) \\
&= e^{iI_{\rm CGB}[g]} \int \mathcal{D}\phi \, e^{iI_\phi[g,\phi]},
\end{aligned}
$$

where the total free energy consists of $F = F^{(0)} + F^{(1)}$.
Note that the tree level free energy $F^{(0)}$ is trivial at the critical point as seen in the previous section, so that the nontrivial contribution to the free energy should come from the one loop effective action.

Effective action

(8) The one loop partition function $Z^{(1)}$:

$$Z^{(1)}[g] = \int \mathcal{D}\phi \, e^{iI_\phi} = \det{}^{-1/2}(-\Box + m^2),$$

(9) The effective action W_ϕ:

$$W_\phi = \frac{i}{2} \ln \det(-\Box + m^2) = \frac{i}{2} \mathrm{Tr} \ln(-\Box + m^2) = \frac{i}{2} \int \frac{d^4x d^4k}{(2\pi)^4} \ln(k_\mu k^\mu + m^2),$$

where k_μ is the conjugate momentum of x^μ.

Local frames

- Note that a (covariant) Fourier transform in curved spacetimes has not been established.
- However, the manifold can be split into a number of small pieces, in which we can consider a Riemann normal coordinates, i.e. $\int_{\mathcal{M}} d^4x\sqrt{-g} \simeq \sum_{U\subset\mathcal{M}} \int_U d^4\tilde{x}$, where $\tilde{x}$ represents the Riemann normal coordinates.
- Then, one can perform the calculation in the momentum space by using the Fourier transform, $-\tilde{\Box} + m^2 \to \tilde{k}_\mu \tilde{k}^\mu + m^2$, where $\tilde{k}$ is the momentum measured in the local coordinates.
- Consequently, it is possible to recover the global coordinates for the covariant result.

Semiclassical Euclidean action formulation cont'd

(10) The Euclidean one loop effective action at the finite temperature:

$$W_\phi = \frac{1}{2}\sum_n \int \frac{d^3x d^3k}{(2\pi)^3} \ln\left(\frac{4\pi^2 n^2}{f\beta^2} + E_m^2\right),$$
$$= \beta \int \frac{d^3x d^3k}{(2\pi)^3}\left[\frac{\sqrt{f}E_m}{2} + \frac{1}{\beta}\ln\left(1 - e^{-\beta\sqrt{f}E_m}\right)\right].$$

where $\sum_n \ln\left(\frac{4\pi^2 n^2}{\beta} + E_m^2\right) = 2\beta\left[\frac{E_m}{2} + \frac{1}{\beta}\ln\left(1 - e^{-\beta E_m}\right)\right]$,

$$E_m^2 \equiv g^{ij}k_i k_j + m^2 = f\, k_r^2 + \frac{k_\theta^2}{r^2} + \frac{k_\phi^2}{r^2\sin^2\theta} + m^2.$$

(NB)

(i) The first term is independent of the black hole temperature.

(ii) The temperature dependent free energy is $\beta F^{(1)} = W_\phi$.

trick

(11) The one loop free energy:

$$F^{(1)} = \frac{1}{\beta} \int_{-\infty}^{\infty} d\omega \int \frac{d^3x d^3k}{(2\pi)^3} \delta(\omega - \sqrt{f} E_m) \ln\left(1 - e^{-\beta\omega}\right)$$
$$= -\int_0^{\infty} d\omega \frac{1}{e^{\beta\omega} - 1} \int_{V_p} \frac{d^3x d^3k}{(2\pi)^3},$$

since the delta function can be written as the derivative of a step function, $\delta(x) = \frac{d}{dx}\epsilon(x)$, defining the step function as $\epsilon(x) = 1$ for $x > 0$ and 0 for $x < 0$. Here, V_p is the volume of the phase space satisfying $\sqrt{f} E_m \leq \omega$, which can be explicitly written as

$$f k_r^2 + \frac{k_\theta^2}{r^2} + \frac{k_\phi^2}{r^2 \sin^2\theta} \leq \frac{\omega^2}{f} - m^2.$$

one loop free energy

(12)The number of quantum states with energy less than ω:

$$n(\omega) \equiv \int_{V_p} \frac{d^3x d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3} \int_{V_p} dr d\theta d\phi dk_r dk_\theta dk_\phi$$

(13)The one loop free energy (= the brick wall free energy)

$$F^{(1)} = -\int d\omega \, \frac{n(\omega)}{e^{\beta\omega} - 1}.$$

Outline

The number of quantum states with energy less than ω for a spherically symmetric black hole:

$$n(\omega) = \frac{2}{3\pi} \int dr \frac{r^2}{\sqrt{f}} \left(\frac{\omega^2}{f} - m^2 \right)^{3/2}.$$

(14) The free energy:

$$\begin{aligned} F^{(1)} &= -\frac{2}{3\pi} \int dr \, \frac{r^2}{f^2} \int_{m\sqrt{f}}^{\infty} d\omega \, \frac{\left(\omega^2 - m^2 f\right)^{3/2}}{e^{\beta\omega} - 1} \\ &\approx -\frac{2\pi^3 r_h^4}{45\beta^4 \epsilon} (1 - \Lambda r_h^2)^{-2} \end{aligned}$$

in the leading order.

(NB)

(i) r_h, ϵ are the event horizon of the black hole and the UV cutoff parameter

(ii) ϵ is assumed to be very small compared to the event horizon with a condition $m^2 \ll r_+/[\epsilon\beta^2(1 - \Lambda r_+^2)]$.

(15)The proper length for the UV cutoff parameter:

$$\bar{\epsilon} \equiv \int_{r_h}^{r_h+\epsilon} dr \sqrt{g_{rr}} \approx \frac{2\sqrt{r_h \epsilon}}{\sqrt{1-\Lambda r_h^2}},$$

which is independent of the parameters of the black hole.

(16)The free energy:

$$F^{(1)} = -\frac{8\pi^3 r_h^5}{45\beta^4 \bar{\epsilon}^2}(1-\Lambda r_h^2)^{-3}.$$

(17) The entropy:

$$S^{(1)} = \beta^2 \frac{\partial F^{(1)}}{\partial \beta}\bigg|_{\beta=\beta_H} = \frac{32\pi^3 r_h^5}{45\beta_H^3 \bar{\epsilon}^2}(1-\Lambda r_h^2)^{-3}.$$

(18) The energy:

$$E^{(1)} = F^{(1)} + \beta^{-1} S^{(1)}\big|_{\beta=\beta_H} = \frac{8\pi^3 r_h^5}{15\beta_H^4 \bar{\epsilon}^2}(1-\Lambda r_h^2)^{-3}.$$

(19)The entropy

$$S^{(1)} = \frac{\ell_p^2}{90\pi\bar{\epsilon}^2}\frac{c^3\mathcal{A}}{4G\hbar}$$

by recovering dimensions and plugging the Hawking temperature into Eq. (17).

(NB)

(i)$\mathcal{A} = 4\pi r_h^2$ the area of horizon

(ii)$\ell_p = \sqrt{G\hbar/c^3}$ Plank length

The entropy agrees with the Bekenstein-Hawking entropy $S^{(1)} = c^3\mathcal{A}/(4G\hbar)$ when the cutoff is chosen as $\bar{\epsilon} = \ell_p/\sqrt{90\pi}$ which is exactly same as in the case of the Schwarzschild black hole. ['t Hooft '85]

(NB)

(20)The free energy and the energy:

$$F^{(1)} = -\frac{c^4 r_h}{16G}(1-\Lambda r_h^2) = -\frac{c^3\mathcal{A}}{16G\hbar\beta_H},$$

$$E^{(1)} = \frac{3c^4 r_h}{16G}(1-\Lambda r_h^2) = \frac{3c^3\mathcal{A}}{16G\hbar\beta_H}.$$

(21) The heat capacity:

$$C_V^{(1)} \equiv T_H \frac{\partial S^{(1)}}{\partial T_H} = T_H \left(\frac{\partial S^{(1)}}{\partial r_h}\right)\left(\frac{\partial T_H}{\partial r_h}\right)^{-1}$$
$$= -\frac{c^3 \mathcal{A}}{2G\hbar}\left(\frac{1-\Lambda r_h^2}{1+\Lambda r_h^2}\right),$$

which is positive for $r_h > 1/\sqrt{|\Lambda|}$ and negative otherwise. It means that the SAdS black hole is stable for large black holes and unstable for small black holes.

(NB)

For $\Lambda = 0$, the heat capacity is always negative, which coincides with the thermodynamic stability of the Schwarzschild black hole.

Outline

- Conclusion

  The total free energy consists of the tree and the one loop correction $F = F^{(0)} + F^{(1)}$, which yield the total entropy $S = S^{(0)} + S^{(1)}$. In spite of the vanishing tree entropy $S^{(0)} = 0$, the total entropy gives the area law from the semi-classical one-loop correction.

- Discussions
  So far, we have assumed that both the scalar decoupling condition and the critical condition are valid on the fixed background. But, one may wonder whether these conditions are still met or not when one considers the one loop back reaction of the geometry because it may affect the classical geometry.
  (i) The bare action:
  $$I[g] = \frac{1}{16\pi G_B} \int d^4x\sqrt{-g}\left[R - 2\Lambda_B + \alpha_B R_{\mu\nu}R^{\mu\nu} + \beta_B R^2\right],$$
  with $\alpha_B = \alpha + \hbar\delta\alpha$, $\beta_B = \beta + \hbar\delta\beta$, and $G_B^{-1} = G^{-1} + \hbar\delta G^{-1}$.
  (NB) At the tree level,
  $\alpha + 3\beta = 0$ (scalar decoupling condition)
  $\alpha - 3/2\Lambda = 0$ (critical condition)
- Now, the one loop effective action of the scalar field with mass m can be also written in the form of the divergent higher curvature terms so that the renormalization yields
  $\alpha_R/G_R = \alpha_B/G_B + \hbar A/120\pi$
  $\beta_R/G_R = \beta_B/G_B + \hbar A/240\pi$,

- Note that $A \approx \ln(\mu^2/m^2) + 2\ln(2/3) + O(m^2/\mu^2)$ is a divergent constant for $\mu \to \infty$. [Birrell & Davies '82, Demers, Lafrance & Myers '95]
- It means
  $\alpha_R + 3\beta_R = (\hbar G_R/24\pi)\ln(2/3)$
  $\alpha_R - 3/2\Lambda_R =$
  $(\hbar G_R/480\pi\Lambda^2)[4(2\Lambda^2 + 60\Lambda m^2 - 45m^4)\ln(2/3) + 15m^2(8\Lambda - 9m^2)]$
  by appropriate counter terms.
- Fortunately, by rescaling $\mu \to 3\mu/2$, one can still require $\alpha_R + 3\beta_R = 0$ in order to avoid the existence of the scalar graviton; however, the critical condition is still violated as $\alpha_R - 3/2\Lambda_R = (\hbar G_R/32\pi\Lambda^2)m^2(8\Lambda - 9m^2)$.
- Therefore, one cannot maintain the scalar graviton decoupling condition and the critical condition simultaneously.