Electromagnetic duality from integrable spin chain

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Seminar Talk
ITP CAS

Gauge/Bethe correspondence

Deep connections between SUSY theories and integrable models

Minahan-Zarembo

Operator dimension in $\mathcal{N} = 4$ SYM $\Longleftrightarrow$ Spin chain spectrum

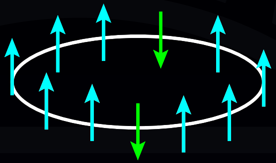
Nekrasov-Shatashvili

Vacua of $\mathcal{N} = 2$ theories $\Longleftrightarrow$ Eigenstate of integrable models

Plan of the talk

- Quantum integrable models
  - Spin chain
  - Toda chain
- Seiberg-Witten theory
  - Pure $SU(N_c)$
  - Superconformal QCD $N_f = 2N_c$
- Nekrasov-Shatashvili quantization
  - Ω background
  - A/B quantization
- Electromagnetic duality as particle-hole duality

Spin chain



- Periodic lattice of L spin sites
- Hamiltonian acts on $V = \overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{L}$

$$\mathcal{H} = \sum_{k=1}^{L}(1 - \mathcal{P}_{k,k+1}), \qquad \mathcal{P}\,|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle \text{ Permutation}$$

- Vacuum: $|0\rangle = |\uparrow\uparrow\ldots\rangle, \qquad \mathcal{H}|0\rangle = 0$
- 1 magnon state: $|\ell\rangle = |\ldots\uparrow\overset{\ell}{\downarrow}\uparrow\ldots\rangle$, but not an eigenstate

Hamiltonian is a $2^L \times 2^L$ matrix, hard to diagonalise!

Bethe ansatz

- 1-magnon eigenstate: $|p\rangle = \sum_{\ell} e^{ip\ell}|\ell\rangle$
- 2-magnon eigenstates:

$$|p_1, p_2\rangle = \sum_{\ell_2 > \ell_1} e^{(ip_1\ell_1 + ip_2\ell_2)}|\ell_1, \ell_2\rangle + S(p_1, p_2)\ e^{(ip_2\ell_1 + ip_1\ell_2)}|\ell_1, \ell_2\rangle$$

$$= |\ldots \uparrow \underset{\ell_1}{\overset{\overrightarrow{p_1}}{\downarrow}} \uparrow \ldots \uparrow \underset{\ell_2}{\overset{\overleftarrow{p_2}}{\downarrow}} \uparrow \ldots\rangle + S(p_1, p_2)\ |\ldots \uparrow \underset{\ell_1}{\overset{\overleftarrow{p_2}}{\downarrow}} \uparrow \ldots \uparrow \underset{\ell_2}{\overset{\overrightarrow{p_1}}{\downarrow}} \uparrow \ldots\rangle$$

Eigenstate when $S(p_1, p_2) = \dfrac{e^{ip_1+ip_2} + 1 - 2e^{ip_1}}{e^{ip_1+ip_2} + 1 - 2e^{ip_2}}$

Integrability

- Periodicity $\Longrightarrow$ quantisation conditions for magnon momenta

$$e^{ip_k L} = \prod_{\ell \neq k}^{M} S(p_\ell, p_k)$$

- Rapidity $x_k = \frac{1}{2} \cot \frac{p_k}{2}$

$$\left(\frac{x_k + i/2}{x_k - i/2} \right)^L = \prod_{\ell \neq k}^{M} \frac{x_k - x_\ell - i}{x_k - x_\ell + i}$$

- Generalize: inhomogeneity θ_k, spin s_k, twisted boundary q

$$\left(\frac{x_k - \theta_k - is_k}{x_k - \theta_k + is_k} \right)^L = q \prod_{\ell \neq k}^{M} \frac{x_k - x_\ell + i}{x_k - x_\ell - i}$$

- Heisenberg spin chain is integrable!
  i.e. L commuting conserved charges $[H_i, H_j] = 0$

One slide proof (Faddeev's train trick)

1. Define Lax matrix $L_{i0}(x) = x\mathbb{I}_{i0} + iJ_{i0}$
   (i: spin site, 0: auxiliary space)

   satisfying Yang-Baxter equation

$$L_{i0}(x)L_{i0'}(x-y)L_{00'}(y) = L_{00'}(y)L_{i0'}(x-y)L_{i0}(x)$$

2. Construct monodromy matrix, also satisfies Yang-Baxter

$$T_0(x) = L_{L0}L_{(L-1)0}\cdots L_{10}(x)$$

3. Transfer matrix $t(x) = \mathrm{tr}_0 T_0(x)$ commute: $[t(x), t(y)] = 0$

Algebraic Bethe ansatz

- Goal: diagonalize $t(x)$
- Let's write Lax, monodromy and transfer **matrices**

$$L_{i0} = \begin{pmatrix} x + iJ_i^3 & iJ_i^+ \\ iJ_i^- & x - iJ_i^3 \end{pmatrix}, \; T_0(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}, \; t(x) = A(x) + D(x)$$

- Vacuum $C(x)|0\rangle = 0, \;\; t(x)|0\rangle = \Big[a(x) + d(x)\Big]|0\rangle$
- Excited state $|x_1, \ldots, x_M\rangle = B(x_1)\cdots B(x_M)|0\rangle$ is an eigenstate of $t(x)$ if Baxter tQ relation holds

$$a(x)Q(x+i) + d(x)Q(x-i) = t(x)Q(x), \quad Q(x) = \prod_{k=1}^{M}(x - x_k)$$

Dual Bethe ansatz for holes

Baxter equation for inhomogeneous chain with twisted boundary

$$-ha(x)Q(x+i) + (h+2)d(x)Q(x-i) = t(x)Q(x)$$

$$a(x) = \prod_{k=1}^{L}(x - \theta_k + is_k), \quad d(x) = \prod_{k=1}^{L}(x - \theta_k - is_k)$$

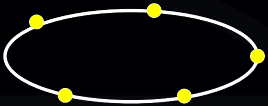
- Magnons: zeros x_k of $Q(x)$, satisfy the Bethe ansatz

$$\left(\frac{x_k - \theta_k - is_k}{x_k - \theta_k + is_k}\right)^L = q \prod_{\ell \neq k}^{M} \frac{x_k - x_\ell + i}{x_k - x_\ell - i}, \quad q = -\frac{h}{h+2}$$

- Holes: zeros ϕ_k of $t(x)$, satisfy the dual Bethe ansatz

$$q = \prod_{\ell=1}^{L} \frac{\Gamma(1 + i(\phi_k - \phi_\ell)}{\Gamma(1 - i(\phi_k - \phi_\ell)} \frac{\Gamma(s_\ell - i(\phi_k - \theta_\ell))}{\Gamma(s_\ell + i(\phi_k - \theta_\ell))} \left[1 + \mathcal{O}(q)\right]$$

Toda chain



- Periodic lattice of L particles interacting via exponential potential

$$\mathcal{H}(x_1, \ldots, x_L) = \sum_{i=1}^{L} \frac{p_i^2}{2} + e^{x_1 - x_2} + \cdots + e^{x_{L-1} - x_L} + \Lambda^2 e^{x_L - x_1}$$

- Baxter equation same as spin chain when $\theta_k - is_k \to \infty$

$$Q(x-i) + \Lambda^{2L} Q(x+i) = t(x) Q(x)$$

- Q not polynomial, no magnons, only holes
- It is also the Seiberg-Witten curve of $SU(L)$ SYM

Pure $\mathcal{N} = 2$ SYM in 4D

- $\mathcal{N} = 2$ vector multiplet: $\mathcal{N} = 1$ chiral multiplet Φ_i + vector multiplet $W_{\alpha i}$

$$\mathcal{L} = \mathrm{Im}\left(\int d^4\theta \, \frac{\partial \mathcal{F}}{\partial \Phi_i}\Phi_i^\dagger + \frac{1}{2}\int d^2\theta \, \frac{\partial^2 \mathcal{F}}{\partial \Phi_i \partial \Phi_j} W_i^\alpha W_{\alpha j}\right)$$

- Potential $V = \mathrm{tr}[\phi, \phi^\dagger]^2$
- If $\phi \neq 0$, $SU(N_c) \xrightarrow{\text{Higgs}} U(1)^{N_c - 1}$, $\quad \phi = \begin{pmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_{N_c} \end{pmatrix}$
- Vacua parametrized by Coulomb moduli $\mathrm{tr}\phi^2, \ldots, \mathrm{tr}\phi^{N_c}$

IR Lagrangian fixed by prepotential $\mathcal{F}$, coupling $\tau_{ij} = \mathrm{Im}\, \dfrac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}$

Electromagnetic duality

- Magnetic variable $\vec{a}_D = \frac{\partial \mathcal{F}}{\partial \vec{a}}, \qquad \tau_{ij} = \frac{\partial a_{D,i}}{\partial a^j}$
- Legendre transform $\tau W^2 + W_D \cdot W$
- $\tau W^2 \longrightarrow \tau_D W_D^2$ where $\tau_D \tau = -1$
- $(\vec{a}, \vec{a}_D) \longrightarrow (\vec{a}_D, -\vec{a})$

Seiberg-Witten solution

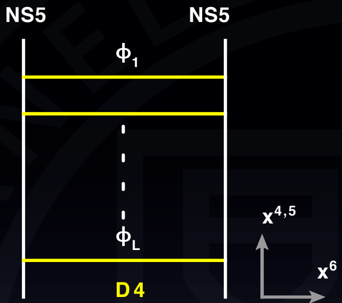
$\mathcal{F}$ determined from meromorphic differential λ_{SW} on genus $N_c - 1$ curve with branch points at $\phi_1, \ldots, \phi_{N_c}$

$$\vec{a} = \frac{1}{2\pi i}\oint_{\vec{A}} \lambda_{SW}, \qquad \vec{a}_D = \frac{1}{2\pi i}\oint_{\vec{B}} \lambda_{SW}, \qquad \vec{m} = \frac{1}{2\pi i}\oint_{\vec{C}} \lambda_{SW}$$

- Seiberg-Witten curve

$$y^2 = \prod_{i=1}^{N_c}(x - \phi_i) - \Lambda^{2N_c}$$

Pure Seiberg-Witten as Toda Chain



- SW curve: $t^2 - t \cdot 2\prod_{i=1}^{N_c}(u-\phi_i) + \Lambda^{2N_c} = 0$
- Let $(u, \log t) = (x, p)$ be conjugate variables, SW curve is now

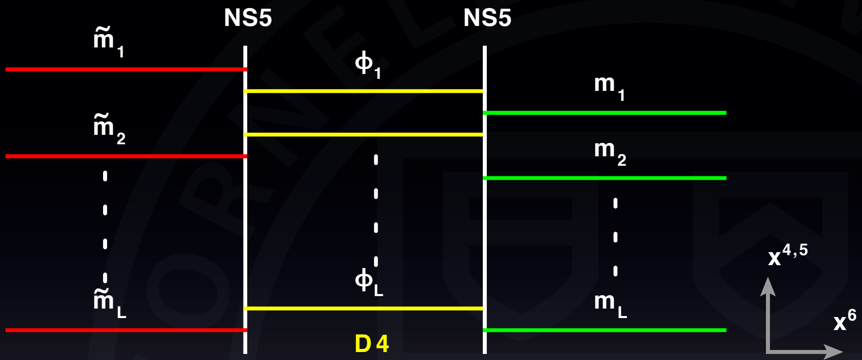
$$e^{2p} - e^p t_L(x) + \Lambda^{2N_c} = 0$$

- Quantize $p \longrightarrow -i\hbar\partial_x$ and act on wave function $Q(x)$

$$Q(x-i\hbar) + \Lambda^{2N_c} Q(x+i\hbar) = t_L(x)Q(x)$$

- Toda chain with $L = N_c$ particles!

Seiberg-Witten with flavor as Spin Chain



$$\prod_{i=1}^{L}(u-\tilde{m}_i)t^2 - t\cdot 2\prod_{i=1}^{L}(u-\phi_i) - h(h+2)\prod_{i=1}^{L}(u-m_i) = 0$$

- Baxter equation for spin chain!

$$-ha(x)Q(x+i\hbar) + (h+2)d(x)Q(x-i\hbar) = t_L(x)Q(x)$$

Dictionary of Gauge/Bethe correspondence

Gauge theory	Spin chain
• Gauge group $U(L)$	Length L of spin chain
• Fundamental mass m_ℓ	Inhomogeneity $\theta_\ell - is_\ell\hbar$
• Anti-fundamental mass $\widetilde{m}_\ell$	Inhomogeneity $\theta_\ell + is_\ell\hbar$
• Coupling $q = e^{2\pi i\tau}$	Twisted boundary condition q
• Coulomb branch parameter ϕ_ℓ	Hole rapidity ϕ_ℓ
• ?	Magnon rapidity x_ℓ

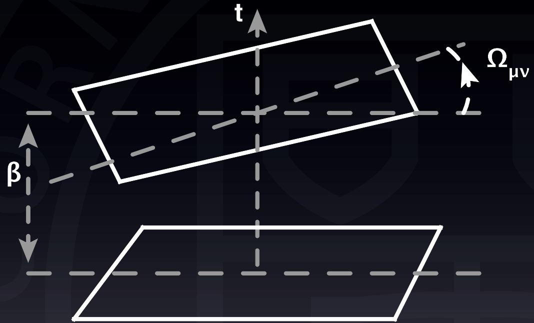
Nekrasov-Shatashvili

When 4D gauge theory is subject to Ω-deformation in one-plane, SUSY vacua becomes quantized and is labelled by magnons.

... and by holes?

Ω background

- Glue two $\mathbb{R}^4 \subset \mathbb{R}^5$ with rotation $\Omega_{\mu\nu} = \begin{pmatrix} & \epsilon_1 & & \\ -\epsilon_1 & & & \\ & & & \epsilon_2 \\ & & -\epsilon_2 & \end{pmatrix}$



- Chemical potential $\mathcal{Z} = \lim_{\beta\to 0} \operatorname{tr} \exp(-\beta H + \epsilon_1 J_{12} + \epsilon_2 J_{34})$
- 4D theory on S^4, lifts Coulomb branch
- Breaks SUSY, localizes the partition function

Nekrasov-Shatashvili deformation

- Deformation only in one plane ($\epsilon_2 = 0$). $\epsilon = \epsilon_1$

${Q^i}_\alpha, {\overline{Q}^i}_{\dot\alpha}$	$SU(2)_{12}$	$SU(2)_{34}$	$U(1)_R$	$SO(4)+U(1)_R$
${Q^1}_1$	1	0	1	1
${Q^1}_2$	-1	0	1	0 Q_+
${Q^2}_1$	1	0	1	1
${Q^2}_2$	-1	0	1	0 Q_-
${\overline{Q}^1}_{\dot 1}$	0	1	-1	0 $\overline{Q}_+$
${\overline{Q}^1}_{\dot 2}$	0	-1	-1	1
${\overline{Q}^2}_{\dot 1}$	0	1	-1	0 $\overline{Q}_-$
${\overline{Q}^2}_{\dot 2}$	0	-1	-1	1

- $\mathcal{N} = 2$ in 4D $\to \mathcal{N} = (2,2)$ in 2D $\quad \{Q_\pm, \overline{Q}_\pm\} = 2(H \pm P)$
- Lifts vaccum, but leaving isolated points
- Twisted chiral field $\overline{D}_+\Sigma = 0, \quad D_-\Sigma = 0$

Two quantization conditions

Low energy theory determined by twisted superpotential

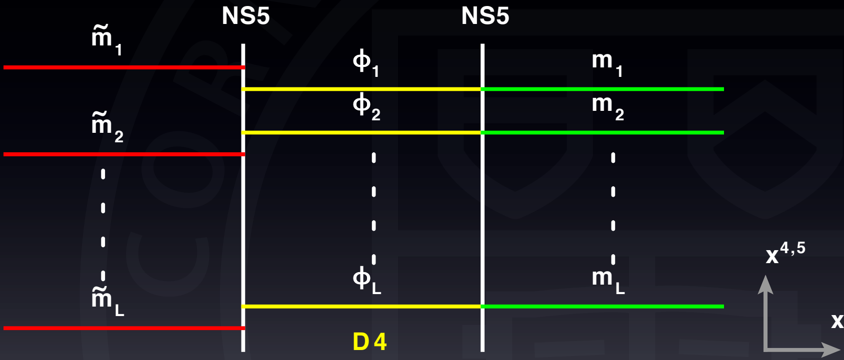
$$\mathcal{F}(\vec{a},\epsilon) = \lim_{\epsilon_2\to 0} \epsilon_1\epsilon_2 \log \mathcal{Z}(\vec{a},\epsilon_1,\epsilon_2)\big|_{\epsilon_1=\epsilon}$$

$$\mathcal{W}(\vec{a},\epsilon) = \frac{1}{\epsilon}\mathcal{F}(\vec{a},\epsilon) - 2\pi i \vec{n}\cdot\vec{a}$$

- A-quantization $\dfrac{\partial \mathcal{W}}{\partial \vec{a}} = 0 \quad \Longrightarrow \quad \vec{a}_D = \vec{n}\epsilon$
- $\vec{n}$ choice of vacuum angle = quantized electric flux F_{03} in $\mathbb{R}^2$
- $\mathbb{Z}_2$ electromagnetic duality $\mathcal{F}_D(\vec{a}_D) = \mathcal{F}(\vec{a}) - \vec{a}\cdot\vec{a}_D$
- B-quantization $\dfrac{\partial \mathcal{W}_D}{\partial \vec{a}_D} = 0 \quad \Longrightarrow \quad \vec{a} = \vec{m} - \vec{n}\epsilon$
- Turning on magnetic flux F_{12} in the undeformed plane

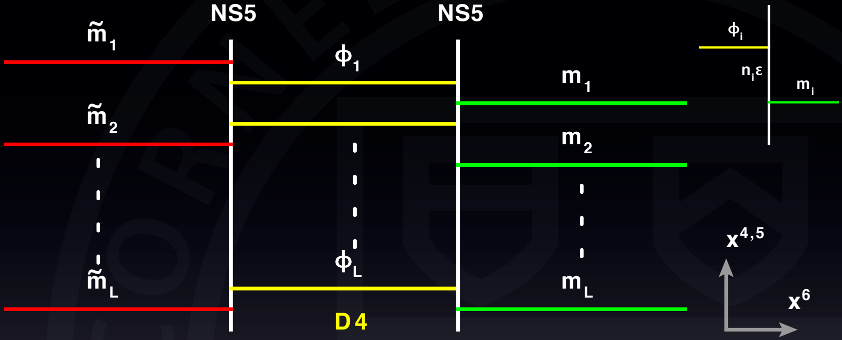
B-quantization

- Root of baryonic Higgs branch $\Longleftrightarrow$ SW curve factorize

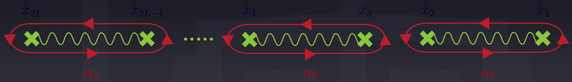


$$\overbrace{-h\,a(x) + (h+2)\,d(x) = t_L(x)}^{\text{Vacuum}} \Leftrightarrow \overbrace{\left[a(x)\,t - (h+2)\,d(x)\right]}^{\text{NS5 + D4 branes}}\,\overbrace{(t+h)}^{NS5'} = 0$$

B-quantization as magnons



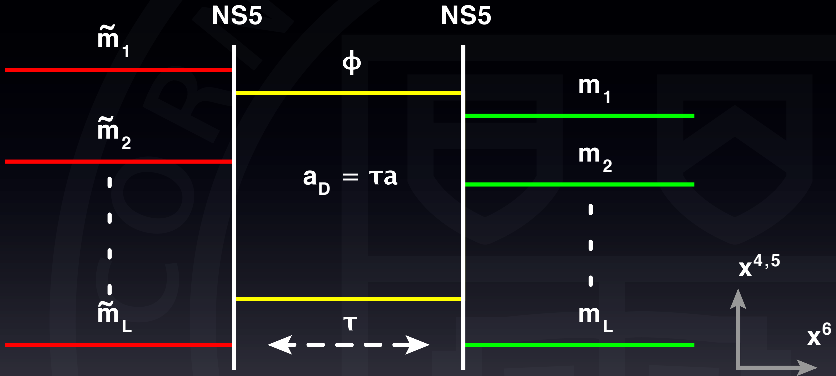
- $\frac{i}{\hbar}\lambda_{SW} = d \log Q$



- Magnons form cuts, quantization condition $\oint_{\vec{\alpha}} \lambda_{SW} = 2\pi\vec{n}\hbar$
- Vacuum $\vec{n} = 0 \overset{\text{Higgs branch root}}{\Longleftrightarrow} \vec{\alpha} = \vec{A} - \vec{C} \Longrightarrow \vec{a} = \vec{m} - \vec{n}\epsilon$

A-quantization as holes

- Difficult to visualize A-quantization on branes. Very roughly,



- We conjecture they correspond to dual Bethe ansatz for holes

A-quantization $\left(\frac{\partial \mathcal{W}}{\partial \vec{a}} = 0\right)$

Compute $\mathcal{W}$ from Nekrasov partition function

$$\mathcal{Z} = \mathcal{Z}_{\text{classical}} \mathcal{Z}_{\text{1-loop}} \mathcal{Z}_{\text{instanton}} \Longrightarrow \mathcal{W} = \mathcal{W}_{\text{classical}} + \mathcal{W}_{\text{1-loop}} + \mathcal{W}_{\text{instanton}}$$

- $\mathcal{W}_{\text{classical}} = -\frac{2\pi i \tau}{\epsilon} \sum_{\ell=1}^{L} a_\ell^2$
- $\mathcal{W}_{\text{1-loop}} = \sum_{\ell,m} \left[-\omega_\epsilon(a_\ell - a_m) + \omega_\epsilon(a_\ell - m_m) + \omega_\epsilon(-a_\ell + \widetilde{m}_m - \epsilon) \right]$

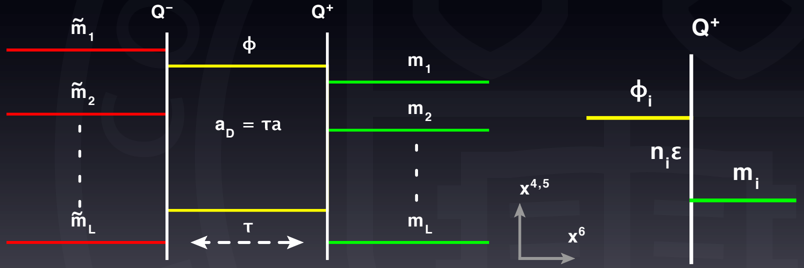
A-quantization: $\qquad \omega'_\epsilon(x) = -\log \Gamma(1 + x/\epsilon)$

$$1 = q^{-\frac{a_\ell}{\epsilon}} \prod_k \frac{\Gamma\left(1 + \frac{a_\ell - a_k}{\epsilon}\right)}{\Gamma\left(1 - \frac{a_\ell - a_k}{\epsilon}\right)} \frac{\Gamma\left(-\frac{a_\ell - \widetilde{m}_k}{\epsilon}\right)}{\Gamma\left(1 + \frac{a_\ell - m_k}{\epsilon}\right)}$$

$$\exp\left[-\int \frac{dy}{2\pi i} \left(\frac{1}{a_\ell - y} + \frac{1}{a_\ell - y - \epsilon} \right) \log\Big(1 + qR(y)Y(y)\Big) \right]$$

A-quantization from spin chain

- Construct meromorphic solutions $Q^{\pm}$ to the Baxter equation
- Require entire solution Q be linear combination of $Q^{\pm}$
- A-quantization $=$ poles of Q^{+} cancel with poles of Q^{-}
- B-quantization $= Q^{+}$ is entire, i.e. poles cancelled by zeros



Is A = B?

- Compare ϕ in A/B in q expansion $\phi = \phi_k^{(0)} + q\phi_k^{(1)} + \cdots$
- $\phi_k^{(0)} = M_\ell - \hat{n}_\ell \epsilon$

$$\phi_k^{(1)} = \frac{1}{\epsilon}\left[R\left(\phi_k^{(0)} - \frac{\epsilon}{2}\right) - R\left(\phi_k^{(0)} + \frac{\epsilon}{2}\right)\right], \quad R(x) = \frac{a(x)d(x+\epsilon)}{t(x)t(x+\epsilon)}$$

- B-quantization same as requiring $Q(x)$ be polynomial
- Analytic properties of $Q^{\pm}$ – Baxter equation is ambiguous!
- Derive A-quantization from counting function?

Thank you!