Pseudo-Supergravity

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Supersymmetry and supergravities work only in certain special cases (such as dimensions $D \leq 11$), and depend upon very specific properties of spinor representations in those dimensions. However, many of the nice features that supersymmetry confers, such as BPS bounds on the energy of a solution, the existence of particularly simple BPS solutions, stability arguments, etc., do not depend upon the entire edifice of exact closure of a super-algebra; the existence of fermionic symmetries described by Killing spinors is sufficient. Here, we report on recent work in this area, including the construction of pseudo-supersymmetric extensions of the effective action for bosonic string in all dimensions.

Based on work with Hong Lü and Zhao-Long Wang.

Uses for Supersymmetry–E.g. $D = 11$ Supergravity

- $D = 11$ supergravity is of the form

$$e^{-1}\mathcal{L} = R - \frac{1}{48}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} + F \wedge F \wedge A + \tfrac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho + \cdots$$

(omitting quartic fermions). Supersymmetry is

$$\begin{aligned} \delta\psi_\mu &= D_\mu \epsilon + \frac{1}{12}\Big(F^{\nu\rho\sigma\lambda}\Gamma_{\mu\nu\rho\sigma\lambda} - \frac{1}{12}F_{\mu\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Big)\epsilon\,, \\ \delta e_\mu^a &= \tfrac{1}{4}\bar{\psi}_\mu \Gamma^a \epsilon\,, \\ \delta A_{\mu\nu\rho} &= \bar{\psi}_{[\mu}\Gamma_{\nu\rho]}\epsilon\,. \end{aligned}$$

where again we have omitted higher-order fermion terms.

- Associated with the supersymmetry is a super-algebra, which is a fermionic extension of the P_μ and $M_{\mu\nu}$ Poincaré algebra with supercharges Q_α satisfying anticommutation relations

$$\{Q, Q\} = C\Gamma^\mu P_\mu + C\Gamma^{\mu\nu} Z_{\mu\nu} + C\Gamma^{\mu_1\cdots\mu_5} Z_{\mu_1\cdots\mu_5}$$

- From this one derives energy inequalities (Bogo'molny bounds)

$$M \geq |Z_{12}|\,, \qquad \text{or} \qquad M \geq |Z_{12345}|$$

where equality saturated for supersymmetric states $Q_\alpha|\rangle = 0$ which have mass equal to central charge (membrane or 5-brane charge).

Membrane and 5-Brane Solutions

- At the field theory level, these BPS states correspond to bosonic solutions of $D = 11$ supergravity for which $\delta\psi_\mu = 0$. Requiring Killing Spinors satisfying

$$D_\mu \epsilon + \frac{1}{12}\Big(F^{\nu\rho\sigma\lambda}\Gamma_{\mu\nu\rho\sigma\lambda} - \frac{1}{8}F_{\mu\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Big)\epsilon = 0$$

  gives equations involving only first-order derivatives of $g_{\mu\nu}$ and $A_{\mu\nu\rho}$, rather than the second-order derivatives in the equations of motion.

- For example, the membrane ansatz

$$ds_{11}^2 = e^{2A(\vec{y})}\, dx^\mu dx_\mu + e^{2B(\vec{y})}\, d\vec{y}\cdot d\vec{y}\,, \qquad A_{\mu\nu\rho} = f\epsilon_{\mu\nu\rho}$$

  gives first-order equations for A, B and f, giving rise to the BPS multi-membranes of Duff & Stelle:

$$ds_{11}^2 = H^{-2/3}\, dx^\mu dx_\mu + H^{1/3}\, d\vec{y}\cdot d\vec{y}\,, \qquad A_{\mu\nu\rho} = H^{-1}\,\epsilon_{\mu\nu\rho}$$

  where H is any harmonic function on the eight-dimensional transverse space of the $\vec{y}$ coordinates.

Consistent Reduction on S^7

- Supersymmetry, and the existence of Killing spinors, also play an important role in the consistency of the S^7 reduction of $D = 11$ supergravity. The Killing spinors of the $\mathrm{AdS}_4 \times S^7$ vacuum played a central role in the proof of consistency by de Wit & Nicolai.

- The S^7 reduction is one of the few examples of a consistent Pauli sphere reduction, where the fields in the reduced theory include the gauge bosons of the isometry group of the sphere.

- A key test: Reduction ansatz for the $SO8$ gauge fields:

$$
\begin{aligned}
d\hat{s}_{11}^2 &= ds_4^2 + (dy^m + K_I^m\, A_\mu^I dx^\mu)^2\,, \\
\hat{F}_{\mu\nu\rho\sigma} &= 3m\epsilon_{\mu\nu\rho\sigma}\,, \qquad \hat{F}_{\mu\nu mn} = F_{\mu\nu}^I\, \nabla_{[m}K_{n]}^I
\end{aligned}
$$

Plug into the $D = 11$ field equations:

$$
\begin{aligned}
& R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - 12m^2 g_{\mu\nu} = \\
& (K_{I\,m} K_J^m + \frac{1}{m^2} \nabla_m K_{I\,n} \nabla^m K_J^n)(F_{\mu\rho}^I F_\nu^{J\rho} - \tfrac{1}{4} F_{\rho\sigma}^I F^{J\,\rho\sigma} g_{\mu\nu})
\end{aligned}
$$

which only make sense if the Killing vectors K_I^m on S^7 satisfy the "miraculous" identity

$$
K_{I\,m} K_J^m + \frac{1}{m^2} \nabla_m K_{I\,n} \nabla^m K_J^n = \delta_{IJ}
$$

Consistent Pauli Sphere Reductions

- Proof is trivial after writing the Killing vectors in terms of the eight Killing spinors on S^7 satisfying $D_m\eta_i - \frac{i}{2}\, m\, \Gamma_m\eta_i = 0$, as

$$K_I^m \leftrightarrow K_{[ij]}^m = \bar\eta_i \Gamma_m \eta_j$$

- Although it is clearly not a coincidence that almost all the known consistent Pauli sphere reductions occur in supergravity theories, this cannot be the whole story since we also know of the remarkable consistent S^3 and S^{D-3} reductions of the effective theory of the bosonic string,

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{a\phi}F^{\mu\nu\rho}F_{\mu\nu\rho}\,, \qquad a^2 = \frac{8}{D-2}$$

  in *any* dimension D, in which all the $SO(4)$ or $SO(D-2)$ gauge bosons are retained.

- This provides one motivation for suspecting that there may exist a "pseudo-supersymmetry" of the effective action of the bosonic string in any dimensions D, which could be sufficient to explain the consistency of the sphere reductions without requiring the entire edifice of supersymmetry.

- Pseudo-supersymmetry would be a symmetry of an extended action with fermions added, provided one looks only up to quadratic order in fermions.

Pseudo-Supersymmetric Extension of Pure Gravity

- A very simple example is the pseudo-supersymmetric extension of pure D-dimensional gravity. The motivation here is that amongst the Ricci–flat solutions of pure gravity, there is a subset that admits Killing spinors; for example, Calabi-Yau spaces or pp-wave solutions (special holonomy). Inspired by what we know from pure $D = 4$ supergravity, we therefore propose a pure D-dimensional pseudo-supergravity:

$$e^{-1}\mathcal{L} = R + \tfrac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho$$

- Action is invariant under pseudo-supersymmetry:

$$\delta\psi_\mu = D_\mu \epsilon\,, \qquad \delta e_\mu^a = \tfrac{1}{4}\bar{\psi}_\mu \Gamma^a \epsilon\,.$$

  That is to say, the δe_μ^a variation of eR cancels the $\delta\psi_\mu$ variation of the "gravitino" kinetic term.

- The symmetry fails beyond the quadratic order in fermions; that is, it is not possible to cancel the quartic fermion terms coming from the e_μ^a variation of the gravitino kinetic term by adding higher-order fermion corrections (except in the special case $D = 4$ of ordinary supergravity).

- Nonetheless, this extended theory allows us to find pseudo-supersymmetric solutions, namely Ricci-flat spaces where in addition $\delta\psi_\mu = 0$, i.e. special-holonomy spaces.

Irreducible Spinors in D Dimensions

- In dimensions 0, 1, 2, 3 and 4 mod 8, spinors can be Majorana, meaning their Majorana and Dirac conjugates are the same: $\bar\psi = \psi^T C = \psi^\dagger \Gamma_0$. When there is a Majorana representation, $C\Gamma^\mu$ and $C\Gamma^{\mu\nu\rho}$ are *symmetric* in their spinor indices, meaning that fermionic kinetic terms $\frac{1}{2}\bar\psi\Gamma^\mu D_\mu\psi$ and $\frac{1}{2}\bar\psi_\mu\Gamma^{\mu\nu\rho} D_\nu\psi_\rho$ are non-trivial for anticommuting Majorana spinors.

- There are no Majorana representations in dimensions 5, 6 or 7 mod 8. Instead, one can double up the spinors by adding an $Sp(2)$ doublet index, and now $\bar\psi_i = \epsilon^{ij}\psi_j^T C = \psi_i^\dagger\Gamma_0$. In these cases where *symplectic-Majorana* representations exist, $C\Gamma^\mu$ and $C\Gamma^{\mu\nu\rho}$ are instead *antisymmetric*. This is just what is needed to make the kinetic terms $\frac{1}{2}\epsilon_{ij}\bar\psi^i\Gamma^\mu D_\mu\psi^j$ and $\frac{1}{2}\epsilon_{ij}\bar\psi^i_\mu\Gamma^{\mu\nu\rho} D_\nu\psi^j_\rho$ non-trivial for anti-commuting symplectic-Majorana spinors.

- With the understanding that for the symplectic-Majorana cases the notation $\frac{1}{2}\bar\psi\Gamma^\mu D_\mu\psi$ means $\frac{1}{2}\epsilon_{ij}\bar\psi^i\Gamma^\mu D_\mu\psi^j$, etc., the Lagrangian we wrote for simple pseudo-supergravity is applicable in any spacetime dimensions D, with ψ_μ being Majorana or symplectic-Majorana as appropriate.

Irreducible Spinors in D Dimensions

The symmetries of $C\Gamma^{(n)}$, where $\Gamma^{(n)}$ denotes $\Gamma^{\mu_1\cdots\mu_n}$, are summarised in the table. The parameter β characterises the symmetry of Γ_μ; thus

$$\Gamma_\mu^T = \beta\, C\Gamma_\mu C^{-1}$$

The table shows when Majorana (M) or symplectic-Majorana (S-M) representations occur in every dimension.

D mod 8	$C\Gamma^{(0)}$	$C\Gamma^{(1)}$	$C\Gamma^{(2)}$	$C\Gamma^{(3)}$	$C\Gamma^{(4)}$	$C\Gamma^{(5)}$	Spinor	β
0	S	S	A	A	S	S	M	$+1$
	S	A	A	S	S	A	S-M	-1
1	S	S	A	A	S	S	M	$+1$
2	S	S	A	A	S	S	M	$+1$
	A	S	S	A	A	S	M	-1
3	A	S	S	A	A	S	M	-1
4	A	S	S	A	A	S	M	-1
	A	A	S	S	A	A	S-M	$+1$
5	A	A	S	S	A	A	S-M	$+1$
6	A	A	S	S	A	A	S-M	$+1$
	S	A	A	S	S	A	S-M	-1
7	S	A	A	S	S	A	S-M	-1

Pseudo-super Extension of the Bosonic String

- Now we turn to the pseudo-supersymmetric extension of the effective action for the bosonic string in D dimensions. By looking at the $D = 10$ case where we know an actual super extension exists, we are led to propose, in D dimensions,

$$
\begin{aligned}
e^{-1}\mathcal{L}_D &= R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{a\phi}F^{\mu\nu\rho}F_{\mu\nu\rho} + \tfrac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho \\
&\quad + \tfrac{1}{2}\bar{\lambda}\not{D}\lambda + e_1\bar{\psi}_\mu\Gamma^\nu\Gamma^\mu\lambda\partial_\nu\phi + e^{\frac{1}{2}a\phi}[e_2\bar{\psi}_\mu\Gamma^{\mu\nu\rho\sigma\lambda}\psi_\lambda \\
&\quad + e_3\bar{\psi}^\nu\Gamma^\rho\psi^\lambda + e_4\bar{\psi}_\mu\Gamma^{\nu\rho\sigma}\Gamma^\mu\lambda + e_5\bar{\lambda}\Gamma^{\nu\rho\sigma}\lambda]F_{\nu\rho\sigma}
\end{aligned}
$$

with transformation rules

$$
\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon + e^{\frac{1}{2}a\phi}(c_1\Gamma_{\mu\nu\rho\sigma}F^{\nu\rho\sigma} + c_2F_{\mu\nu\rho}\Gamma^{\nu\rho})\epsilon\,, \\
\delta\lambda &= c_3\Big((\Gamma^\mu\partial_\mu\phi)\,\epsilon + \frac{a}{12}\,e^{\frac{1}{2}a\phi}\,\Gamma^{\mu\nu\rho}F_{\mu\nu\rho}\,\epsilon\Big)\,, \\
\delta e_\mu^a &= \tfrac{1}{4}\bar{\psi}_\mu\Gamma^a\epsilon\,, \qquad \text{so} \quad \delta g_{\mu\nu} = \tfrac{1}{2}\bar{\psi}_{(\mu}\Gamma_{\nu)}\,\epsilon\,, \\
\delta\phi &= c_4\,\bar{\epsilon}\lambda\,, \\
\delta A_{\mu\nu} &= \bar{\epsilon}\Big[c_5\,\Gamma_{[\mu}\psi_{\nu]} + c_6\,\Gamma_{\mu\nu}\lambda\Big]\,e^{-\frac{1}{2}a\phi}\,,
\end{aligned}
$$

- There is indeed a solution for the constants e_i and c_i in all dimensions, such that we get a pseudo-supersymmetric action. The e_5 term is absent in $D = 10$, but in general is $e_5 = -(D-10)/(48(D-2))$.

Addition of a Gauge Multiplet

- In ten dimensions, $\mathcal{N}=1$ supergravity can be coupled to an $\mathcal{L}=1$ Maxwell or Yang-Mills muliplet. We can generalise the pseudo-supersymmetric extension of the bosonic string to include a gauge multiplet too, in arbitrary dimensions D.

- For a Maxwell multiplet, with gauge potential A_μ, we add a pseudo-gaugino χ, generalise the 3-form field to $\widetilde{F}_{\mu\nu\rho} = F_{\mu\nu\rho} - \frac{1}{2} A_{[\mu} \partial_\nu A_{\rho]}$, and add to the Lagrangian

$$e^{-1}\mathcal{L}_{\text{gauge}} = -\tfrac{1}{4} e^{\frac{a}{2}\phi} F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2}\bar{\chi} \not{D} \chi + e_6\, e^{\frac{a}{4}\phi}\, F^{\rho\sigma}\, \bar{\chi} \Gamma^\mu \Gamma_{\rho\sigma} \psi_\mu$$
$$+ e_7\, e^{\frac{a}{4}\phi}\, F_{\mu\nu}\, \bar{\chi} \Gamma^{\mu\nu} \lambda + e_8\, e^{\frac{a}{2}\phi}\, \widetilde{F}_{\mu\nu\rho}\, \bar{\chi} \Gamma^{\mu\nu\rho} \chi\,.$$

The transformation rules are

$$\delta A_\mu = c_7\, e^{-\frac{a}{4}\phi}\, \bar{\epsilon}\, \Gamma_\mu \chi\,, \qquad \delta\chi = c_8\, e^{\frac{a}{4}\phi}\, F_{\mu\nu}\, \Gamma^{\mu\nu}\, \epsilon$$

Previous transformation rule for $A_{\mu\nu}$ is now augmented by

$$\delta_{\text{extra}}\, A_{\mu\nu} = c_9\, e^{-\frac{a}{4}\phi}\, \bar{\chi} \Gamma_{[\mu} \epsilon\, A_{\nu]}$$

Additionally, $F_{\mu\nu\rho}$ in the original extended string Lagrangian is replaced by $\widetilde{F}_{\mu\nu\rho}$.

- We can indeed solve for the new e_i and c_i coefficients for all dimensions D. Generalisation to Yang-Mills straightforward.

Curvature-squared Corrections

- In string theory, there are α' corrections to the effective action, associated with quadratic and higher curvature terms. It was shown by Bergshoeff and de Roo that the supersymmetrisation of a $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ (up to order α') could be derived by the trick of treating the curvature tensor (with bosonic $F_{\mu\nu\rho}$ torsion) as if it were an $SO(1,9)$ Yang-Mills field. The results for the Yang-Mills multiplet can then be taken over, using the dictionary

$$\begin{aligned} A_\mu &\longrightarrow \omega_{\mu+}^{ab}\,, \\ F_{\mu\nu} &\longrightarrow R_{\mu\nu}{}^{ab}(\omega_+)\,, \\ \chi &\longrightarrow \psi^{ab}\,, \end{aligned}$$

where $\omega_{\mu\pm}^{ab} = \omega_\mu^{ab} \pm \frac{1}{2}F_\mu{}^{ab}$ are connections with torsion, and

$$\psi_{ab} = D_a(\omega_-)\psi_b - D_b(\omega_-)\psi_a$$

is the gravitino curvature.

- The same trick works for our pseudo-supersymmetric generalisations to arbitrary dimensions D.

Conformal Anomaly Term

- Equations of motion for the fields in the bosonic string were derived by calculating beta functions, by Callan, Friedan, Martinec and Perry. In dimensions other than $D = 26$, there is a conformal anomaly that shows up as a dilaton potential term of the form $-\frac{1}{2}m^2 e^{-\frac{1}{2}a\phi}$ in the effective Lagrangian. This can be included in a consistent Pauli S^3 or S^{D-3} reduction. Is it pseudo-supersymmetrisable?

- We find this is indeed possible. The required extra terms in the Lagrangian and transformation rules take the form

$$e^{-1}\mathcal{L}_{\text{conf}} = -\tfrac{1}{2}m^2 e^{-\frac{1}{2}a\phi} + e^{-\frac{1}{4}a\phi}\Big[e_9\,\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu + e_{10}\,\bar{\lambda}\lambda + e_{11}\,\bar{\psi}_\mu\Gamma^\mu\lambda\Big]$$

and

$$\delta_{\text{extra}}\psi_\mu = c_{10}\,e^{-\frac{1}{4}a\phi}\,\Gamma_\mu\epsilon\,,\qquad \delta_{\text{extra}}\lambda = c_{11}\,e^{-\frac{1}{4}a\phi}\,\epsilon\,.$$

- As it stands, this only works for those dimensions and spinor representations where $\beta = -1$, since otherwise $\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu$ and $\bar{\lambda}\lambda$ vanish. For $\beta = +1$ cases, we can double up all the fermions, $\psi_\mu \to \psi_\mu^\alpha$, etc., contract in all previous fermion bilinears with $\delta_{\alpha\beta}$, and in the new terms with $\epsilon_{\alpha\beta}$. (Very like the doubling to symplectic-Majorana when a Majorana representation does not exist.)

Summary and Discussion

- Can pseudo-supersymmetrise the effective action for the bosonic string in any dimension D. A Yang-Mills multiplet can be added in too, thus giving an arbitrary-D (pseudo) generalisation of $\mathcal{N}=1$, $D=10$ Einstein-Yang-Mills supergravity.

- Can also pseudo-supersymmetrise a certain exponential potential term for the dilaton (conformal anomaly). There does not appear to exist a genuinely supersymmetric counterpart for this extension.

- Amongst the solutions of the theories will be bosonic BPS-type solutions, for which there exist Killing spinors ϵ satisfying

$$
\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon + e^{\frac{1}{2}a\phi}\,(c_1\Gamma_{\mu\nu\rho\sigma}F^{\nu\rho\sigma} + c_2F_{\mu\nu\rho}\Gamma^{\nu\rho})\epsilon = 0\,,\\
\delta\lambda &= c_3\Big((\Gamma^\mu\partial_\mu\phi)\,\epsilon + \frac{a}{12}\,e^{\frac{1}{2}a\phi}\,\Gamma^{\mu\nu\rho}F_{\mu\nu\rho}\,\epsilon\Big) = 0\,.
\end{aligned}
$$

  This gives first-order equations, which are easier to solve than the full second-order equations of motion. (Some solutions have been obtained by Haishan Liu, Hong Lü and Zhao-Long Wang.)

- It may be possible to use the pseudo-supersymmetry to gain a better understanding of why the consistent Pauli S^3 and S^{D-3} reductions work.

- If one pushed to higher orders in fermions, at what point would pseudo-supersymmetry fail? Presumably, at the quartic level.

- The pseudo-supersymmetry transformations cannot generate genuine algebras that go beyond the known and classified superalgebras. It is possible that they might generate algebras that violate Jacobi identities. The Jacobi identities for the standard $\mathcal{N} = 1$, $D = 10$ superalgebra depend upon the special gamma-matrix identity

$$g_{\mu\nu}\,\Gamma^{\mu}{}_{(\alpha\beta}\,\Gamma^{\nu}{}_{\gamma\delta)} = 0$$

which holds only in $D = 2$, 4, 6 and 10.

- The fact that the pseudo-supersymmetric extension of the D-dimensional bosonic string exists, and is very non-trivial, suggests there may still be important structures, and symmetries, remaining to be understood.