Schrödinger invariant solutions of M-theory with Enhanced Supersymmetry

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I.T.P.December 25,2009

Motivation and Setup

- AdS/CFT and its applications.

1) AdS/CFT correspondence is a unique approach to strongly coupled field theories in which certain questions become computationally tractable and conceptually more transparent. In condensed matter physics there are many strongly coupled systems that can be engineered and studied in detail in laboratories. It seems reasonable to hope, therefore, that the AdS/CFT correspondence may be able to offer insight into some of these nonconventional materials.

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  2) Condensed matter systems may offer an arena in which many of the fascinating concepts of high energy theory can be experimentally realised. In condensed matter physics there are many effective Hamiltonians. Ultimately one might hope to engineer an emergent field theory with a known AdS dual, thus leading to experimental AdS/CFT.

Motivation and Setup

- AdS/CFT and its applications.

3) AdS/CFT correspondence allows a somewhat rearranged view of nature in which the traditional classification of fields of physics by energy scale is less important. If a quantum gravity theory can be dual to a theory with many features in common with quantum critical electrons, the question of which is more 'fundamental' is not a meaningful question. Instead, the emphasis is on concepts that have meaning on both sides of the duality. This view has practical consequences. For instance, seeking a dual description of superconductivity one realises that there might be loopholes in black hole 'no-hair' theorems and one is led to new types of black hole solutions.

Motivation and Setup

- NR-AdS/CFT

Recently some gravity backgrounds with non-relativistic conformal symmetry were discussed. As it is well known, the DLCQ of a field theory gives a non-relativistic system. If one considers a sector with large non-zero light-cone momentum one can find regions in the geometry where the circle has a non-zero size so that computations can be trusted. For example consider massless Klein-Gordon equation in d dimensional Minkowski spacetime

$$\Box\Phi \equiv -\partial_t^2\Phi + \partial_i^2\Phi = 0 \tag{1}$$

Defining the light-cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}}(t \pm x^d)$

$$-2\partial_+\partial_-\Phi + \partial_i^2\Phi = 0 \tag{2}$$

Identifying $\partial_- = -iM$ then the equation form of the Schrödinger equation in free (d-2)-dimensional space, with light-cone coordinate x^+ playing the role of time

$$\left(2iM\partial_+ + \partial_i^2\right)\Phi = 0 \tag{3}$$

If the parent relativistic theory has a gravity dual one can hope to have a gravity description of the corresponding quantum mechanical system.

Motivation and Setup

- NR-AdS/CFT

Another example is masive scalar theory. Let us begin by Lagrangian of a complex scalar

$$\mathcal{L} = \frac{1}{c^2}\partial_t Z \partial_t \bar{Z} - \partial_i Z \partial_i \bar{Z} - \frac{m^2 c^2}{\hbar^2} Z\bar{Z}, \tag{4}$$

the particle modes are given by

$$Z = \frac{\hbar}{\sqrt{2m}} z\, e^{-imc^2 t/\hbar} \tag{5}$$

Non-relativistic limit is defined by $c \longrightarrow \infty$, in this limit Lagrangian reduce to

$$\mathcal{L}_{NR} = \bar{z}\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_i^2\right) z \tag{6}$$

Motivation and Setup

- ABJM

  $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory with $U(N) \times U(N)$ gauge group with Chern-Simons levels $(k, -k)$. The matter fields consist of bi-fundamental scalars Φ^A and fermions Ψ_A, which transform under the $SU(4) \simeq SO(6)$ R-symmetry group as $\mathbf{4}$ and $\bar{\mathbf{4}}$, respectively. It is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$.

Motivation and Setup

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Motivation and Setup

- ABJM

  $\mathcal{N}=6$ supersymmetric Chern-Simons-matter theory with $U(N)\times U(N)$ gauge group with Chern-Simons levels $(k,-k)$. The matter fields consist of bi-fundamental scalars Φ^A and fermions Ψ_A, which transform under the $SU(4)\simeq SO(6)$ R-symmetry group as $\mathbf{4}$ and $\bar{\mathbf{4}}$, respectively. It is dual to M-theory on $AdS_4\times S^7/\mathbb{Z}_k$.

- NR-ABJM(with 14 supercharges)

  1) performing a mass deformation (which gives the same mass to all matter fields (up to signs for fermions) and breaks the $SU(4)$ R-symmetry into $SU(2)_1\times SU(2)_2\times U(1)_R$.)
  2)takes the usual non-relativistic limit for massive fields.

$$\mathcal{L}=\frac{k}{4\pi}\left(\mathcal{L}_{\rm CS}+\mathcal{L}_{\rm kin}+\mathcal{L}_{\rm bos}+\mathcal{L}_{\rm int1}+\mathcal{L}_{\rm int2}\right),$$

where

Motivation and Setup

- NR-ABJM (with 14 supercharges)

$$
\begin{aligned}
\mathcal{L}_{\rm CS} &= \epsilon^{mnp}\mathrm{tr}\left[A_m\partial_n A_p - \tfrac{2i}{3}A_mA_nA_p - \tilde{A}_m\partial_n\tilde{A}_p + \tfrac{2i}{3}\tilde{A}_m\tilde{A}_n\tilde{A}_p\right] , \\
\mathcal{L}_{\rm kin} &= \mathrm{tr}\left[\bar{\phi}_A(iD_t)\phi^A - (D_i\bar{\phi}_A)(D_i\phi^A)\right] \\
&\quad +\mathrm{tr}\left[\bar{\psi}^A(iD_t)\psi_A + \bar{\psi}^a(D_i^2\psi_a - F_{12}\psi_a + \psi_a\overline{F}_{12}) + \bar{\psi}^{\dot{a}}(D_i^2\psi_{\dot{a}} + F_{12}\psi_{\dot{a}} - \psi_{\dot{a}}\overline{F}_{12})\right] , \\
\mathcal{L}_{\rm bos} &= \tfrac{1}{2}\mathrm{tr}\left[\phi^a\bar{\phi}_{[a}\phi^b\bar{\phi}_{b]} - \phi^{\dot{a}}\bar{\phi}_{[\dot{a}}\phi^{\dot{b}}\bar{\phi}_{\dot{b}]}\right] , \\
\mathcal{L}_{\rm int1} &= \tfrac{1}{4}\mathrm{tr}\left[(\bar{\phi}_a\phi^a + \bar{\phi}_{\dot{a}}\phi^{\dot{a}})(\bar{\psi}^b\psi_b - \bar{\psi}^{\dot{b}}\psi_{\dot{b}}) + (\phi^a\bar{\phi}_a + \phi^{\dot{a}}\bar{\phi}_{\dot{a}})(\psi_b\bar{\psi}^b - \psi_{\dot{b}}\bar{\psi}^{\dot{b}})\right] \\
&\quad + \tfrac{1}{2}\mathrm{tr}\left[-\phi^a\bar{\phi}_b\psi_a\bar{\psi}^b + \phi^{\dot{a}}\bar{\phi}_{\dot{b}}\psi_{\dot{a}}\bar{\psi}^{\dot{b}} - \bar{\phi}_a\phi^b\bar{\psi}^a\psi_b + \bar{\phi}_{\dot{a}}\phi^{\dot{b}}\bar{\psi}^{\dot{a}}\psi_{\dot{b}}\right] , \\
\mathcal{L}_{\rm int2} &= -\tfrac{1}{2}\mathrm{tr}\left[\epsilon^{ab}\epsilon^{\dot{c}\dot{d}}(\bar{\phi}_a\psi_b\bar{\phi}_{\dot{c}}\psi_{\dot{d}} + \bar{\phi}_a\psi_{\dot{c}}\bar{\phi}_{\dot{d}}\psi_b) + \epsilon_{ab}\epsilon_{\dot{c}\dot{d}}(\phi^a\bar{\psi}^b\phi^{\dot{c}}\bar{\psi}^{\dot{d}} + \phi^a\bar{\psi}^{\dot{c}}\phi^{\dot{d}}\bar{\psi}^b)\right] .
\end{aligned}
$$

The Lagrangian (7) is invariant under the scaling

$$(t, x\,;\,\phi,\psi) \;\rightarrow\; (\lambda^{-2}t, \lambda^{-1}x\,;\,\lambda\phi, \lambda\psi)\,.$$

This can be extended to the full Schrödinger algebra which also includes a non-relativistic special conformal symmetry generator K.

Motivation and Setup

- NR-ABJM: Supersymmetry

12 Poincaré supercharges of the ABJM theory survive the mass deformation as well as the non-relativistic limit. (Four of them are singlets under $SU(2)_1 \times SU(2)_2$. Two of them $(Q, \bar{Q})$, which anti-commute to give the Hamiltonian H, are called dynamical. The other two $(q, \bar{q})$ which anti-commute to give the $U(1)_B$ generator are called kinematical. The remaining eight supercharges $\{q_{a\dot{a}}, \bar{q}^{a\dot{a}}\}$ transform in $(\mathbf{2}, \mathbf{2})$ of $SU(2)_1 \times SU(2)_2$, which we call *spectators*, commute with all Schrödinger generators except the rotation.)

Commutators between K and $(Q, \bar{Q})$ require that an additional pair of supercharges $(S, \bar{S})$, called conformal supercharges, should exist.

In summary, the NR-ABJM theory has the global symmetry group

$$U(1)_B \times SU(2)_1 \times SU(2)_2 \times U(1)_R \times \mathbb{Z}_2 \, ,$$

where the $\mathbb{Z}_2$ interchanges the two $SU(2)$ factors, and contains 14 supercharges.

Motivation and Setup

- BW/LLM solution and subtleties with the NR limit:

The gravity dual of the ABJM theory is $AdS_4 \times S^7/\mathbb{Z}_k$. therefore to find the gravity dual of the NR-ABJM theory, one has to carry over the mass deformation and the non-relativistic limit to the gravity side.

The gravity dual of the mass deformed theory was obtained some time ago by Bena and Warner (and reproduced later by Lin, Lunin and Maldacena (LLM)). Bena-Warner begins with a collection of M2-branes and turns on the four-form flux in the transverse directions. The flux breaks the $SO(8)$ R-symmetry to $SO(4) \times SO(4)$ and polarizes the M2-branes into M5-branes, which wrap the two three-spheres that are orbits of the $SO(4)$ groups.

The NR-ABJM theory is non-trivial when there are non-zero number of particles, which is proportional to the eigenvalue of the $U(1)_B$ generator, which in turn gets identified with the central element M of the Schrödinger algebra.

Recall that the $U(1)_B$ generator acts on the circle fiber of S^7. On the other hand, in the geometric realization of the Schrödinger algebra, M is identified with a light-cone momentum.

Motivation and Setup

- BW/LLM solution and subtleties with the NR limit:

The situation is strongly reminiscent of the discrete light-cone quantization (DLCQ) procedure taken in the context of Schrödinger geometry. A crucial difference is that in our case the light-cone momentum is taken along a direction transverse to the M2-brane world-volume. In principle, one could proceed as follows:

First, one modifies the BW/LLM solution by adding the particle number M. In the IIA picture, it amounts to turning on the flux counting the D0-brane charge.

Second, one makes the standard coordinate change of the DLCQ procedure:

$$\begin{aligned} & \tilde{\phi} = \phi - \alpha t\,, \quad \tilde{t} = t \\ \Rightarrow \qquad & \widetilde{H} \equiv i\partial_{\tilde{t}} = i\partial_t - \alpha(-i\partial_\phi) \equiv H - \alpha M\,, \quad \widetilde{M} \equiv -i\partial_{\tilde{\phi}} = -i\partial_\phi \equiv M\,. \end{aligned}$$

With a suitably chosen constant α and an appropriate scaling limit, the light-cone Hamiltonian is identified with the Hamiltonian of the non-relativistic theory. The gravity description is expected to be valid for a large value of M.

Motivation and Setup

- BW/LLM solution and subtleties with the NR limit:

  Unfortunately, we are hindered by a technical difficulty; it is not clear how to turn on the M momentum and obtain the fully back reacted supergravity solution, as the $U(1)_B$ circle is fibered non-trivially along the $\mathbb{CP}^3$ base.

  We are thus led to an alternative approach. We will begin with the most general ansatz consistent with the symmetries of the NR-ABJM theory and look for a supergravity solution preserving the same amount of supersymmetry.

Motivation and Setup

- Super-Schrödinger symmetry (Bosonic part)

The Schrödinger algebra Sch_d contains an $SO(2,1)$ subalgebra among the time-translation (H), dilatation (D) and special conformal (C) generators.

$$[D,H]=+2H\,,\qquad [D,C]=-2C\,,\qquad [H,C]=-D\,,$$

as well as the $SO(d)$ subalgebra,

$$[M^{ij},M^{kl}]=+\delta^{jk}M^{il}+\delta^{il}M^{jk}-\delta^{ik}M^{jl}-\delta^{jl}M^{ik}\,.$$

The remaining generators are space-translations (P^i) and Galilean boosts (G^i). They are vectors under the $SO(d)$,

$$[M^{ij},P^k]=+\delta^{jk}P^i-\delta^{ik}P^j\,,\qquad [M^{ij},G^k]=+\delta^{jk}G^i-\delta^{ik}G^j\,,$$

and satisfy the following commutation relations:

$$\begin{aligned} &&[D,P^i]=+P^i\,,&&[D,G^i]=-G^i\,,&&\\ [H,P^i]=0\,,&&[C,P^i]=+G^i\,,&&[H,G^i]=-P^i\,,&&[C,G^i]=0\,.\end{aligned}\tag{7}$$

Finally, we have the central extension with the “rest-mass” or the particle number,

$$[P^i,G^j]=-\delta^{ij}M\,.$$

Motivation and Setup

- Super-Schrödinger symmetry (Global fram)

It is sometimes useful to introduce a Virasoro-like notation,

$$L_0 \equiv \tfrac{1}{2} D\,,\ \ L_{-1} \equiv H\,,\ \ L_{+1} \equiv C\,,\quad P^i_{-1/2} \equiv P^i\,,\ \ P^i_{+1/2} \equiv G^i\,,\quad M_0 \equiv M\,.$$

Then, the commutation relations can be compactly summarized as

$$[L_m, L_n] = (m-n)L_{m+n}\,,\quad [L_m, P^i_r] = \left(\tfrac{1}{2} m - r\right) P^i_{m+r}\,,\quad [P^i_r, P^j_s] = (r-s)\delta^{ij} M_{r+s}\,.$$

The operator-state map naturally introduces the following recombination of generators:

$$\widehat{L}_0 \equiv \frac{1}{2}(-iH - iC)\,,\quad \widehat{L}_{\pm 1} \equiv \frac{1}{2}(-iH + iC \pm D)\,,$$

$$\widehat{P}^i_{\pm 1/2} = \frac{1}{\sqrt{2}}(-iP^i \mp G^i)\,,\quad \widehat{M}_0 = -iM_0\,.$$

The new generators also satisfy Virasoro-like commutation relations,

$$[\widehat{L}_m, \widehat{L}_n] = (m-n)L_{m+n}\,,\quad [\widehat{L}_m, \widehat{P}^i_r] = \left(\tfrac{1}{2} m - r\right) \widehat{P}^i_{m+r}\,,\quad [\widehat{P}^i_r, \widehat{P}^j_s] = (r-s)\delta^{ij} \widehat{M}_{r+s}\,,$$

as well as the conjugation relations

$$(\widehat{L}_m)^\dagger = L_{-m}\,,\quad (\widehat{P}^i_r)^\dagger = P^i_{-r}\,,\quad (\widehat{M}_0)^\dagger = \widehat{M}_0\,.$$

Motivation and Setup

- Super-Schrödinger symmetry (Geometric realization)

In a $(d+3)$-dimensional case, Schrödinger-invariant metric can be written as

$$ds^2 = -\frac{dt^2}{r^4} + \frac{2dtdv + d\vec{x}^2 + dr^2}{r^2}\,.$$

The generators of the Schrödinger algebra are realized as Killing vectors of this metric,

$$L_m = -t^{m+1}\partial_t - \tfrac{1}{2}(m+1)t^m(r\partial_r + x^i\partial_i) + \tfrac{1}{4}m(m+1)t^{m-1}(\vec{x}^2 + r^2)\partial_v\,,$$
$$P^i_r = t^{r+1/2}\partial^i - (r+\tfrac{1}{2})t^{r-1/2}x^i\partial_v\,, \quad M_m = t^m\partial_v\,, \quad M_{ij} = x_i\partial_j - x_j\partial_i\,.$$

In global the new coordinate, the metric reads

$$ds^2 = -\frac{dT^2}{R^4} + \frac{2dTdV - (\vec{X}^2 + R^2)dT^2 + d\vec{X}^2 + dR^2}{R^2}\,.$$

The global form of the Schrödinger generators get simplified in this coordinate,

$$\widehat{L}_0 = \tfrac{1}{2}(i\partial_T)\,, \quad \widehat{L}_{\pm1} = \tfrac{1}{2}e^{\pm 2iT}\left[i\partial_T + i(\vec{X}^2 + R^2)\partial_V \mp \left(X^i\partial_{X^i} + R\partial_R\right)\right]\,,$$
$$\widehat{P}^i_{\pm1/2} = \tfrac{1}{\sqrt{2}}e^{\pm iT}\left(-i\partial_{X^i} \mp X^i\partial_V\right)\,, \quad \widehat{M}_0 = -i\partial_V\,.$$

Motivation and Setup

- Super-Schrödinger symmetry (Case 2+3)

Let $J \equiv -iM^{12}$ be the $SO(2)$ rotation generator. It is useful to combine other generators according to their helicity (J-eigenvalue) defined by

$$[J, \mathcal{O}] = j\mathcal{O}\,.$$

For example, $P_r \equiv P_r^1 + iP_r^2$ has $j = +1$ and $\bar{P}_r \equiv P_r^1 - iP_r^2$ has $j = -1$. In the helicity basis, the bosonic algebra can be rewritten as

$$[L_m, L_n] = (m-n)L_{m+n}\,, \quad [L_m, P_r] = \left(\tfrac{1}{2}m - r\right)P_{m+r}\,, \quad [P_r, \bar{P}_s] = 2(r-s)M_{r+s}\,.$$

In what follows, we will denote operators with non-negative j by unbarred operators $\mathcal{O}$ and their hermitian conjugates by barred operators $\bar{\mathcal{O}}$.

Motivation and Setup

- $\mathcal{N} = 2$ super-Sch. algebra

The notation $\mathcal{N} = 2$ refers to the supersymmetry of the relativistic parent theory. In the "Poincaré frame", it has kinematical $(q, \bar{q})$, dynamical $(Q, \bar{Q})$ and conformal $(S, \bar{S})$ supercharges, and a $U(1)$ R-symmetry.
In the Virasoro-like notation the supercharges are denoted by q, $Q_{-1/2} \equiv Q$, $Q_{+1/2} \equiv S$ and their conjugates. They transform under the $SO(2,1) \times U(1)_J \times U(1)_R$ subalgebra as

$$[L_m, Q_r] = \left(\tfrac{1}{2}m - r\right) Q_r \,, \quad [L_m, q] = 0 \,,$$

$$[J, Q_r] = +\tfrac{1}{2} Q_r \,, \quad [R, Q_r] = +Q_r \,, \quad [J, q] = +\tfrac{1}{2} q \,, \quad [R, q] = -q \,.$$

$$[\bar{P}_r, Q_s] = (r - s)\bar{q} \,, \qquad [\bar{P}_r, q] = 0 \,,$$

and anti-commutators among supercharges give

$$\{\bar{Q}_r, Q_s\} = L_{r+s} + \tfrac{1}{2}(r - s)\left(J - \tfrac{3}{2}R\right) \,, \qquad \{q, Q_r\} = P_r \,, \qquad \{\bar{q}, q\} = 2M \,.$$

Note that $(L_m, Q_r, J - \frac{3}{2}R)$ form a closed sub-algebra, called $OSp(2|1)$, isomorphic to the usual $\mathcal{N} = 2$ superconformal algebra in a chiral sector of RNS superstring world-sheet.

Motivation and Setup

- $\mathcal{N} = 6$ super-Sch. algebra

The additional eight supercharges, which we call *spectator* supercharges, satisfy the following relations:

$$
\begin{aligned}
&[L_m, q_{a\dot{a}}] = 0\,, \quad [P_r, q_{a\dot{a}}] = 0 = [\bar{P}_r, q_{a\dot{a}}]\,,\\
&\{Q_r, q_{a\dot{a}}\} = 0 = \{\bar{Q}_r, q_{a\dot{a}}\}\,, \quad \{q, q_{a\dot{a}}\} = 0 = \{\bar{q}, q_{a\dot{a}}\}\,,\\
&[J, q_{a\dot{a}}] = +\tfrac{1}{2} q_{a\dot{a}}\,, \quad [R, q_{a\dot{a}}] = 0\,,\\
&[R^a{}_b, q_{c\dot{c}}] = -\delta^\alpha_\gamma q_{b\dot{c}} + \tfrac{1}{2}\delta^a_b q_{c\dot{c}}\,, \quad [R^{\dot{a}}{}_{\dot{b}}, q_{c\dot{c}}] = -\delta^a_c q_{\dot{c}\dot{b}} + \tfrac{1}{2}\delta^{\dot{a}}_{\dot{b}} q_{c\dot{c}}\,,\\
&\left\{\bar{q}^{a\dot{a}}, q_{b\dot{b}}\right\} = \frac{1}{2}\delta^a_b \delta^{\dot{a}}_{\dot{b}} M - \delta^a_b R^{\dot{a}}{}_{\dot{b}} + \delta^{\dot{a}}_{\dot{b}} R^a{}_b\,,
\end{aligned}
$$

where $R^a{}_b$, $R^{\dot{a}}{}_{\dot{b}}$ are the $SU(2)$ generators defined by

$$[R^a{}_b, R^c{}_d] = \delta^c_b R^a{}_d - \delta^a_d R^c{}_b\,, \quad (R^a{}_b)^\dagger = R^b{}_a\,.$$

The $\mathcal{N} = 2$ subalgebra still holds, except that the generator R is replaced by $\tilde{R}(4/3)R - (2/3)\Sigma$. In the field theory, the shift is partly due to an additional conserved quantity, namely, the fermion number Σ. From the commutation relations, we see that the shift is needed to make $q_{a\dot{a}}$ neutral under $J - \frac{3}{2}\tilde{R}$, which should hold because $q_{a\dot{a}}$ commutes with Q_r.

Motivation and Setup

- Ansatz

Recall the sequence of the R-symmetry breaking,

$$SO(8) \supset U(1)_B \times SU(4) \supset U(1)_B \times SU(2)_1 \times SU(2)_2 \times U(1)_R \,.$$

To see how these R-symmetries are realized geometrically, consider S^7 as a warped product of two S^3's, and write down the metric as

$$ds^2_{S^7} = d\alpha^2 + \cos^2\alpha \, d\Omega_1^2 + \sin^2\alpha \, d\Omega_2^2 \,.$$

We use the standard Euler-angle coordinates (θ, ϕ, ψ) for each S^3:

$$d\Omega_i^2 = \frac{1}{4}\left[d\theta_i^2 + \sin^2\theta_i d\phi_i^2 + (d\psi_i - \cos\theta_i d\phi_i)^2\right] \qquad (i = 1, 2, \text{ no sum}).$$

We choose the orientations of the 3-spheres such that the $U(1)_R$ acts diagonally on $\psi_{1,2}$ and the $U(1)_B$ acts with an opposite relative sign.
Now, let us begin with $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ and imagine taking the mass deformation and then the non-relativistic limit. The procedure will change the metric significantly, but the R-symmetries (8) as well as the time and space translation (in Poincaré patch) should be preserved throughout. Moreover, the fibration structure of the $U(1)_B$ and $U(1)_R$ angles over the two S^2's should be maintained.
In what follows, we will use the following notations

$$w = \tfrac{1}{2}(\psi_1 + \psi_2)\,, \quad v = \tfrac{1}{2}(\psi_1 - \psi_2)\,, \tag{8}$$

Motivation and Setup

- Metric

We can try to write down the most general ansatz for the metric and the 4-form flux consistent with the Schrödinger symmetry, global symmetries as well as the fibration structure. we propose our ansatz for the metric,

$$ds^2 = e^{2c_1}\left(-c_2\frac{dt^2}{r^4} + \frac{2dt(Dv + c_3 Dw) + dr^2 + d\vec{x}^2}{r^2} + \frac{4}{9}e^{2h_2}(Dw)^2\right) + e^{-4c_1}\left(e^{-2h_2}dy^2 + \frac{4}{3}e^{2h_1}(e^{+2h_3}d\omega_1^2 + e^{-2h_3}d\omega_2^2)\right).$$

All the functions ($c_{1,2,3}$, $h_{0,1,2,3}$) depend only on y, which is the only coordinate not constrained by the continuous symmetries of the geometry. We “gauge-fixed” the reparametriztion invariance in y by a particular choice of g_{yy}. The numerical factors 4/9 and 4/3 are inserted for later convenience.

(The Schrödinger symmetry and R-symmetry allow for two more terms in the metric, $r^{-2}dtdy$ and $Dwdy$, but both of them can be removed by shifting v and w by y-dependent functions.)

Motivation and Setup

- Orthonormal frame

The metric ansatz admits a natural orthonormal frame,

$$e^+ = \frac{e^{2c_1}}{r^2}\,dt\,, \quad e^- = -\frac{c_2}{2r^2}\,dt + Dv + c_3\,Dw\,,$$
$$e^1 = \frac{e^{c_1}}{r}\,dx^1\,, \quad e^2 = \frac{e^{c_1}}{r}\,dx^2\,, \quad e^7 = \tfrac{2}{3}\,e^{c_1+h_2}\,Dw\,, \quad e^8 = \frac{e^{c_1}}{r}\,dr\,, \quad e^9 = e^{-2c_1-h_2}\,dy\,,$$
$$(e^3, e^4\,;\,e^5, e^6) = \frac{1}{\sqrt{3}}\,e^{-2c_1+h_1}\left(e^{+h_3}(\sigma_1,\sigma_2)\,;\,e^{-h_3}(\tau_1,\tau_2)\right).$$

Here, σ_A, τ_A are invariant one forms of S^3's.

Motivation and Setup

- Flux

$$
\begin{aligned}
F \quad = \quad & e^{-3c_1}\,e^{+8}\left[e^{-2c_1}\,k_1\,e^{12}+e^{4c_1-2h_1}\left(e^{-2h_3}\,k_{4,1}\,e^{34}+e^{+2h_3}\,k_{4,2}\,e^{56}\right)\right] \\
& +e^{h_2}\,e^{+9}\left[e^{-2c_1}\,k_2\,e^{12}+e^{4c_1-2h_1}\left(e^{-2h_3}\,k_{5,1}\,e^{34}+e^{+2h_3}\,k_{5,2}\,e^{56}\right)\right] \\
& +e^{c_1}\,e^{97}\left[e^{-3c_1}\,k_3\,e^{+8}+e^{4c_1-2h_1}\left(e^{-2h_3}\,k_{6,1}\,e^{34}+e^{+2h_3}\,k_{6,2}\,e^{56}\right)\right] \\
& +e^{8c_1-4h_1}\,k_7\,e^{3456}\,.
\end{aligned}
$$

Here, we are using the shorthand notation $e^{ab}=e^a\wedge e^b$, etc. and assuming wedge products among differential forms. We inserted compensating factors of metric coefficients so that the Bianchi identity ($dF=0$) maintains the simple form,

$$
k_1'+4k_2=0\,,\quad k_{4,1}'+2k_{5,1}-k_3=0\,,\quad k_{4,2}'+2k_{5,2}-k_3=0\,,\quad k_7'-(k_{6,1}+k_{6,2})=0\,. \tag{9}
$$

Motivation and Setup

- Flux

$$
\begin{aligned}
F \quad = \quad & e^{-3c_1}\,e^{+8}\left[e^{-2c_1}\,k_1\,e^{12}+e^{4c_1-2h_1}\,(e^{-2h_3}\,k_{4,1}\,e^{34}+e^{+2h_3}\,k_{4,2}\,e^{56})\right]\\
& +e^{h_2}\,e^{+9}\left[e^{-2c_1}\,k_2\,e^{12}+e^{4c_1-2h_1}\,(e^{-2h_3}\,k_{5,1}\,e^{34}+e^{+2h_3}\,k_{5,2}\,e^{56})\right]\\
& +e^{c_1}\,e^{97}\left[e^{-3c_1}\,k_3\,e^{+8}+e^{4c_1-2h_1}\,(e^{-2h_3}\,k_{6,1}\,e^{34}+e^{+2h_3}\,k_{6,2}\,e^{56})\right]\\
& +e^{8c_1-4h_1}\,k_7\,e^{3456}\,.
\end{aligned}
$$

Here, we are using the shorthand notation $e^{ab}=e^a\wedge e^b$, etc. and assuming wedge products among differential forms. We inserted compensating factors of metric coefficients so that the Bianchi identity ($dF=0$) maintains the simple form,

$$
k_1'+4k_2=0\,,\quad k_{4,1}'+2k_{5,1}-k_3=0\,,\quad k_{4,2}'+2k_{5,2}-k_3=0\,,\quad k_7'-(k_{6,1}+k_{6,2})=0\,. \tag{9}
$$

- Parity symmetry There is a discrete $\mathbb{Z}_2$ symmetry exchanging the two 2-spheres which acts as a parity $y\rightarrow -y$. The unknown functions have the following parity eigenvalues,

$$
\begin{aligned}
\text{Even} \quad &: \quad c_1,c_2,h_1,h_2,k_1,(k_{4,1}+k_{4,2}),(k_{5,1}-k_{5,2}),(k_{6,1}+k_{6,2})\,.\\
\text{Odd} \quad &: \quad c_3,h_3,k_2,k_3,(k_{4,1}-k_{4,2}),(k_{5,1}+k_{5,2}),(k_{6,1}-k_{6,2}),k_7\,.
\end{aligned}
$$

Schrödinger invariant solutions of M-theory with Enhanced Supersymmetry

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I.T.P. December 19, 2009

Solution and a Sketch of the Computation

- Methods

Our approach to the problem will hinge upon two standard tools used for finding supersymmetric solutions, namely, the spinorial Lie derivative and the G-structure. Lie derivative of a spinor ϵ with respect to a Killing vector K may be defined as

$$\mathfrak{L}_K \epsilon = K^m \nabla_m \epsilon + \frac{1}{4} (\nabla_a K_b) \Gamma^{ab} \epsilon.$$

In general, the spinorial Lie derivative gives a geometric realization of the algebra,

$$[K, Q_1] = Q_2 \quad \Longleftrightarrow \mathfrak{L}_K \epsilon_{Q_1} = \epsilon_{Q_2}\,.$$

From the metric ansatz, one may then write out the spinoral Lie derivatives associated to the various Killing directions. The Lie derivatives of the spinors, via the super Schrödinger algebra discussed in section 2, determine all coordinate dependence other than the y-direction of the two dynamical supercharges Q. Once Q are determined, the kinematical q and conformal S supercharges also may be worked out from the algebra. Assuming the existence of Killing spinors $\{\epsilon_i\}$, one constructs the following differential forms

$$\begin{aligned} K_{ij} &= (\bar{\epsilon}_i \Gamma_a \epsilon_j) e^a\,, \\ \Omega_{ij} &= \frac{1}{2} (\bar{\epsilon}_i \Gamma_{ab} \epsilon_j) e^{ab}\,, \\ \Sigma_{ij} &= \frac{1}{5!} (\bar{\epsilon}_i \Gamma_{abcde} \epsilon_j) e^{abcde}\,. \end{aligned}$$

Solution and a Sketch of the Computation

- Methods

The Killing spinor equations imply that K_{ij} are Killing vectors, so that (10) becomes a geometric representation of the algebra

$$\{Q_i, Q_j\} = K_{ij}\,.$$

In addition, the KSE give a set of algebraic and differential relations among (K, Ω, Σ). These relations are equivalent to the original KSE by construction, but are often easier to solve and illuminate the geometric structure more clearly.
We will demand that our ansatz admit the six supercharges of $\mathcal{N} = 2$ super-Sch algebra. The kinematical supercharges $(q, \bar{q})$ correspond to null Killing spinors whereas the dynamical supercharges $(Q, \bar{Q})$ correspond to time-like Killing spinors. We first focus on the real combination $\epsilon = \frac{1}{2}(q + \bar{q})$ which satisfies the two projection conditions

$$\Gamma^{3456}\epsilon = -\epsilon \;\; (\text{singlet under } SU(2)_1 \times SU(2)_2)\,, \quad \Gamma^{+}\epsilon = 0\,,$$

Restoring both components $(q, \bar{q})$ then defines an $SU(4)$ sub-structure of the Spin(7) structure. Having started by introducing an ansatz, making the G-structure manifest entails a small frame rotation from the original frame to the canonical G-structure frame. Similarly, for $(Q, \bar{Q})$ we find an $SU(4)$ sub-structure of the $SU(5)$ structure. The conformal supercharges $(S, \bar{S})$ do not yield any new information because they are related to $(Q, \bar{Q})$ by the conformal symmetry generator and all bosonic symmetries are already

Solution and a Sketch of the Computation

- Equations

Block A : The equations for (c_1, h_1, h_2, h_3) decouple from all other variables.

$$4h_1' - h_2' = -c_1'(2h_1' + h_2')^2 e^{6c_1+2h_2}\,, \qquad 9c_1' = (9c_1' - 4h_1' + h_2')e^{2h_2}\,,$$
$$2h_1' + h_2' = 6(h_1' + h_3')e^{-6c_1+2h_1-2h_2+2h_3}\,, \quad h_3' \cosh(2h_3) = -h_1' \sinh(2h_3)\,.$$

The following auxiliary equations will also be useful,

$$\cos\zeta = e^{h_2}\,, \qquad \sin\zeta = -\tfrac{1}{3}(2h_1' + h_2')e^{3c_1+2h_2} = \frac{1}{3c_1'}(-\zeta' \cos\zeta + 2e^{-3c_1})\,.$$

Solution and a Sketch of the Computation

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Block B : With the solutions of Block A as an input, we can solve the equations for (c_3, k_1, k_2, k_3).

$$k_2 = -k_3\,, \quad k_1 = -\frac{6c_3}{\sin\zeta}e^{3c_1}\,, \quad 3c_3' = 2\left(k_1 e^{-6c_1}\frac{h_1' - h_2'}{2h_1' + h_2'} - k_3 e^{-3c_1}\sin\zeta\right)$$
$$3c_3' + k_1 e^{-6c_1} = 6\sin\zeta(c_3\cosh(2h_3) - \sinh(2h_3))e^{3c_1-2h_1}\,,$$

Solution and a Sketch of the Computation

- Equations

Block C : The last metric component c_2 and all the remaining flux components are determined algebraically by the solutions of Block A and Block B.

$$
\begin{aligned}
c_2 &= \left(\tfrac{1}{4}\,k_1\,e^{-3c_1}\right)^2\,,\\
k_{4,1} &= -\tfrac{3}{2}(c_3+1)e^{3c_1}\sin\zeta - \tfrac{1}{4}\,k_1\left(2e^{-6c_1+2h_1+2h_3} - e^{2h_2}\right),\\
k_{4,2} &= -\tfrac{3}{2}(c_3-1)e^{3c_1}\sin\zeta - \tfrac{1}{4}\,k_1\left(2e^{-6c_1+2h_1-2h_3} - e^{2h_2}\right),\\
k_{5,1} &= -\tfrac{3}{2}(c_3-1)e^{+4h_2}\,,\\
k_{5,2} &= -\tfrac{3}{2}(c_3+1)e^{-4h_2}\,,\\
k_{6,1} &= -\frac{h_1'+2h_2'+3h_3'}{3(h_1'+h_3')}\,e^{2h_2}\,,\\
k_{6,2} &= -\frac{h_1'+2h_2'-3h_3'}{3(h_1'-h_3')}\,e^{2h_2}\,,\\
k_7 &= 6c_1'\,e^{-6c_1+4h_1}\,.
\end{aligned}
$$

Solution and a Sketch of the Computation

- Solution

  The final form of the solution may be most neatly captured in terms of two quadratic polynomials,

$$g_1 = 1 - y^2\,, \quad g_2 = 1 + \tfrac{1}{2}cy + y^2\,.$$

The metric components are

$$\begin{array}{lll} e^{6c_1} = g_1^2 g_2^{-1}\,, & c_2 = b^2 g_1^{-2} g_2^{-1}\,, & c_3 = \tfrac{4}{3} byg_1^{-2}\,, \\ e^{2h_1} = g_1\,, & e^{2h_2} = 1 - 4y^2 e^{-6c_1}\,, & e^{2h_3} = 1\,, \end{array}$$

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and the flux components are

$$\begin{aligned} & k_1 = -4bg_2^{-1}\,, \quad k_2 = -bg_2' g_2^{-2}\,, \quad k_3 = bg_2' g_2^{-2}\,, \\ & k_{4,1} = -3y + b(2g_1^{-1} - g_2^{-1}) \quad & & k_{4,2} = +3y + b(2g_1^{-1} - g_2^{-1})\,, \\ & k_{5,1} = +\tfrac{3}{2} - 2ybg_1^{-2} \quad & & k_{5,2} = -\tfrac{3}{2} - 2ybg_1^{-2}\,, \\ & k_{6,1} = k_{6,2} = 1 - 4g_2 g_1^{-2}\,, \quad & & k_7 = -4g_2' g_1^{-1} + 2g_1' + 3g_2'\,. \end{aligned}$$

Solution and a Sketch of the Computation

- Non-existence of spectator supercharges

  Our original goal was to find the gravity dual of the NR-ABJM theory with 14 supercharges. But, the Killing spinor equations for the six $\mathcal{N} = 2$ supercharges have already determined all unknown functions in our ansatz completely. Proceeding with the same methods, it is not difficult to show that our solution does not admit the other eight 'spectator' supercharges.

Discussion

- No-spectator

We saw in section 2 that the anti-commutations of two spectator supercharges give the generators for $SU(2)_1 \times SU(2)_2$ as well as the central element M.

$$\left\{\bar{q}^{a\dot{a}}, q_{b\dot{b}}\right\} = \frac{1}{2}\delta^a_b\delta^{\dot{a}}_{\dot{b}}M - \delta^a_b R^{\dot{a}}{}_{\dot{b}} + \delta^{\dot{a}}_{\dot{b}} R^a{}_b\,,$$

From the geometric point of view, the spinor bi-linears $\bar{\epsilon}\Gamma^m\epsilon$ made of the Killing spinors $\epsilon_{a\dot{a}}$ corresponding to $q_{a\dot{a}}$ should produce the Killing vectors for the generators on the right hand side of above expression. Now, recall that $q_{a\dot{\alpha}}$ commute with $(H, D, C, P, \bar{P}, M)$. Inspecting the spinorial Lie derivatives, especially $\mathfrak{L}_C\epsilon$, we find that $\epsilon_{a\dot{a}}$ must be annihilated by Γ^+. This implies that all bi-linears constructed from $\epsilon_{a\dot{a}}$ can have non-zero components only in the (x^-)-directions much like the kinematical supercharges $(q, \bar{q})$ discussed earlier:

$$\bar{\epsilon}^{a\dot{a}}\Gamma^m\epsilon_{b\dot{b}} = 0 \quad (\text{except for } m = -)\,.$$

In particular, the generators for $SU(2)_1 \times SU(2)_2$ symmetry cannot be produced by the Killing spinors. We thus proved without much computation that the Killing spinors for the spectator supercharges with desired algebraic property do not exist within our ansatz.

Discussion

- No-spectator

Even if we give up the $SU(2)_1 \times SU(2)_2$ generators, it is still impossible to obtain eight extra Killing spinors as one can see from the following counting argument. We argued above for the projection condition $\Gamma^+\epsilon_{a\dot{a}} = 0$. The fact that $q_{a\dot{a}}$ transform in the same way under the two $SU(2)$ groups imply that $\partial_v\epsilon_{a\dot{a}} = 0$, which together with $\mathfrak{L}_M\epsilon_{a\dot{a}} = 0$ yield another projection condition, $\Gamma^{3456}\epsilon_{a\dot{a}} = -\epsilon_{a\dot{a}}$. Finally, since $\epsilon_{a\dot{a}}$ are null Killing spinors, it enforces yet another condition, $\Gamma^9\epsilon_{a\dot{a}} = \epsilon_{a\dot{a}}$. Three mutually orthogonal projection conditions leave at most $32/2^3 = 4$ independent components, so the possiblity of eight extra spinors is excluded.

Discussion

- No-spectator

  We have shown that a supergravity background dual to the NR-ABJM theory preserving the super-Schrödinger symmetry and all the global symmetries does not exist. We do not have a clear physical understanding of why this is the case. We end this talk with two possible directions we may pursue to find an explanation.

Discussion

- No-spectator

We have shown that a supergravity background dual to the NR-ABJM theory preserving the super-Schrödinger symmetry and all the global symmetries does not exist. We do not have a clear physical understanding of why this is the case. We end this talk with two possible directions we may pursue to find an explanation.

First, it is conceivable that the singularity problem of the unpolarized BW/LLM solution mentioned in section 2 is unavoidable, so that even if we find a good way to take the non-relativistic limit, the resulting geometry would be necessarily singular. If this is true, we may need to doubt either the existence of the NR-ABJM theory as a quantum field theory or the validity of non-relativistic holography.

Discussion

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We have shown that a supergravity background dual to the NR-ABJM theory preserving the super-Schrödinger symmetry and all the global symmetries does not exist. We do not have a clear physical understanding of why this is the case. We end this paper with two possible directions we may pursue to find an explanation.

First, it is conceivable that the singularity problem of the unpolarized BW/LLM solution mentioned in section 2 is unavoidable, so that even if we find a good way to take the non-relativistic limit, the resulting geometry would be necessarily singular. If this is true, we may need to doubt either the existence of the NR-ABJM theory as a quantum field theory or the validity of non-relativistic holography.

Second, note that we have searched for a gravity solution preserving all Schrödinger and global symmetries apart from the non-zero particle number (M-eigenvalue). Via holography, it would correspond to a ground state of the NR-ABJM theory for a fixed non-zero particle number that preserves all the symmetries. It is not obvious a priori whether such a ground state should exist in the field theory. If holography works, the non-existence of the fully symmetric gravity solution may be an indication that the ground states of the field theory necessarily break some parts of the symmetries.