A commutant of $\beta\gamma$-System associated to the highest weight module V_4 of $sl(2, \mathbb{C})$

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Abstract: Analogue to the commutant in the theory of associative algebras, one can construct a new subalgebra of vertex algebra known as vertex algebra commutants. In this paper, for the highest weight module V_4 of Lie algebra $sl(2, \mathbb{C})$, we describe a commutant of $\beta\gamma$-System $S(V_4)$ by giving its finite generators and OPE relations among generators.
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1 Introduction

Let W be a vertex algebra, and U be its subalgebra, one can construct a new subalgebra, which is known as the commutant $Com(U, W)$ of U in W. In fact, this construction is a generalization of the coset construction in conformal field theory due to Kac-Peterson [2] and Goddard-Ken-Olive [3], and was introduced by I. Frenkel and Zhu in [4] in mathematics. The construction is analogue to the ordinary commutant construction in associative algebra theory.

In classical invariant theory, the description of associative algebra commutants is an important problem. At present, there have been many new computational methods promoted by the developments of commutative algebra and algebraic geometry. In some how, a vertex algebra is a generalized associative algebra. So we believe that the commutants in vertex algebra should be an important object as in associative algebra.

In order to describe vertex algebra commutant $Com(U, W)$ more clearly, we expect to find its generators set and the corresponding relations. But we don't know whether the commutant $Com(U, W)$ is generated finitely, and how to find its generators. These are non-trivial problems.

It's obviously that $U \subset Com(Com(U, W), W)$. If $U = Com(Com(U, W), W)$, the pair U and $(Com(U, W))$ is called a Howe pair([14]). As in associate

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algebra, it should have applications in vertex algebra theory. Since up to now, we know little about these new objects, it should be useful to construct new more examples in the vertex algebra category.

According to [5], the notion of a commutative circle algebra is abstractly equivalent to the notion of a vertex algebra, and there are the related correspondence between two theories. We shall refer to a commutative circle algebra simply as a vertex algebra throughout this paper.

Let $\mathbf{g}$ be a finite-dimensional, complex semisimple Lie algebra, V be a finite-dimensional complex $\mathbf{g}-$module via $\rho : \mathbf{g} \longrightarrow EndV$, there is a vertex algebra $S(V)$ known as $\beta\gamma-$system(was induced in [1]), the representation ρ induces a vertex algebra homomorphism $\widehat{\rho}$ from $\mathcal{O}(\mathbf{g}, B)$ to $S(V)$, where B is the bilinear form $B(u, v) = -Tr(\rho(u)\rho(v))$. If V admits a symmetric invariant bilinear form B', there is a vertex algebra homomorphism $\widehat{\psi}$: $\mathcal{O}(sl(2, \mathbb{C}), -\frac{dimV}{8}K) \longrightarrow S(V)^{\Theta_+}$, denote by $\mathcal{A} = \widehat{\psi}(\mathcal{O}(sl(2, \mathbb{C}), -\frac{n}{8}K))$(cf. §2 in [8] in details). In [14], Zhu introduced a functor which assigns each vertex algebra W to an associative algebra $A(W)$(known as the Zhu algebra of W). From [4], we know that the associative algebra $A(\mathcal{O}(\mathbf{g}, B))$ is isomorphic to the universal enveloping algebra $U(\mathbf{g})$. Let $D(V)$ be the Weyl algebra of polynomial differential operators of V. It is well known that $A(S(V))$ is isomorphic to $D(V)$. In classical invariant theory, Schwartz studied the commutants of $D(V)$ in [7]. Analogous to the classical commutant theory of associative algebras, we would like to describe the commutant $S(V)^{\Theta_+}$ of $\Theta = \widehat{\rho}(\mathcal{O}(\mathbf{g}, B))$ in $S(V)$ by giving its generators and relations.

In terms of descriptions of the commutant $S(V)^{\Theta_+}$, there have been some important results. In [8, 11], B. Lian and Linshaw studied the vertex algebra and invariant theory, and reduced the problem of describing $S(V)^{\Theta_+}$ to a problem in commutative algebra. They single out a certain category $\mathfrak{R}$ of vertex algebras equipped with a $\mathbb{Z}_{\geq 0}-$ filtration such that the associated graded objects are $\mathbb{Z}_{\geq 0}-$graded $\partial-$ rings. All vertex algebras of the form $S(V)$ and $\mathcal{O}(\mathbf{g}, B)$ belong to the category $\mathfrak{R}$, and so are their subalgebras $\widehat{\rho}(\mathcal{O}(\mathbf{g}, B))$ and $S(V)^{\Theta_+}$. In particular, the assignment $W \longmapsto gr(W)$ is a functor from $\mathfrak{R}$ to the category $\mathcal{R}$ of $\mathbb{Z}_{\geq 0}-$ graded $\partial-$rings, and the main object of study $S(V)^{\Theta_+}$ is sent to the $\partial-$ ring $gr(S(V)^{\Theta_+})$. It's lucky that the reconstruction property of the category $\mathcal{R}$ tells us that if one can find a generator set of $gr(S(V)^{\Theta_+})$, then he can construct a generator set of $S(V)^{\Theta_+}$. However, describing generators of $gr(S(V))^{\Theta_+}$ is much easier than that of $gr(S(V)^{\Theta_+})$ in the invariant theory. Moreover, there is a canonical injection $\Gamma : gr(S(V)^{\Theta_+}) \longrightarrow gr(S(V))^{\Theta_+}$, if Γ is surjective, the generator set of $gr(S(V))^{\Theta_+}$ can be regarded as the generator set of $gr(S(V)^{\Theta_+})$, hence, one need to find the generator set of $gr(S(V))^{\Theta_+}$. As an example in [11], taking $\mathbf{g} = V = sl(2, \mathbb{C})$, Linshaw studied the subalgebras $gr(S_\beta(V)^{\Theta_+})$ and $gr(S_\gamma(V)^{\Theta_+})$ of $gr(S(V)^{\Theta_+})$, and gave a complete description of vertex algebras $S_\beta(V)^{\Theta_+}$ and $S_\gamma(V)^{\Theta_+}$, moreover, he showed that $\mathcal{A}$ is isomorphic to the current algebra $\mathcal{O}(sl(2, \mathbb{C}), -\frac{3}{8}K)$. In terms of $V = \mathbf{g} = sl(2, \mathbb{C})$, in [8], the authors used the properties of Gröbner bases to prove that $S(V)^{\mathcal{A}_+} = \Theta$ and obtained a Howe pair $(\Theta, S(V)^{\Theta_+})$ in $S(V)$. About the case that $\mathbf{g}$ is abelian Lie algebra acting diagonally on a vector space V, Linshaw found a finite set of generators for $S(V)^{\Theta_+}$, and showed that $S(V)^{\Theta_+}$ is a simple vertex algebra and a member of Howe pair ([10]). More generally, if $\mathbf{g} = sl(n, \mathbb{C}), so(n, \mathbb{C}), sp(2n, \mathbb{C})$ and V are the copies of standard representations of $\mathbf{g}$, Linshaw and Bailin Song used tools from commutative algebra and algebraic geometry, in particular, the theory of jet schemes, to describe $S(V)^{\Theta_+}$ and gave some Howe pairs in vertex algebra context([12]).

In classical invariant theory, Hilbert series is an effective tool to describe an invariant ring, since it can show the shapes of generators of the invariant ring and the relations on generators([13]). Based on the theory of vertex algebras and $\partial-$ rings in [5, 8, 9, 11, 12], we also study the case of $V = \mathbf{g} = sl(2, \mathbb{C})$. Under the related properties of Hilbert series, we find all finite generators of invariant ring $gr(S(V))^{\Theta_+}$, and then we obtain the finite generator set of $S(V)^{\Theta_+}$, moreover, we get a new Howe pair $(\mathcal{A}, S(V)^{\mathcal{A}_+})$ in $S(V)$([15]).

In this paper, we study the case of V_4 which is a highest weight $sl(2, \mathbb{C})-$ module with the highest weight 4. Based on the same methods with [15], we give a finite generator set of the commutant $S(V_4)^{\Theta_+}$. It's more difficult than the case of the adjoint representation of $sl(2, \mathbb{C})$. All the difficulties come from the following, 1. The calculation of the Hilbert series of $V_4 \oplus V_4^*$; 2. $gr(S(V_4))_0^{sl(2,\mathbb{C})}$ having more generators with higher degrees than that of adjoint representation of $V = sl(2, \mathbb{C})$([15]); 3. Finding vertex operators in $S(V_4)^{\Theta_+}$ corresponding to each generator of $gr(S(V_4))_0^{sl(2,\mathbb{C})}$.

Here is an outline of the paper. Firstly, the action of $sl(2, \mathbb{C}) \otimes \mathbb{C}[t]$ on $gr(S(V_4))$ induced by the representation V_4, forms the invariant subalgebra $gr(S(V_4))^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$. Denote by $P = gr(S(V_4))$, using the theorem 5.9 in [12], we know $\partial-$ ring $P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by $P_0^{sl(2,\mathbb{C})}$. In particular, the finite generator set of $P_0^{sl(2,\mathbb{C})}$ is also the generator set of $P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ as a $\partial-$ ring. Secondly, we calculate the Hilbert series of $V_4 \oplus V_4^*$, thereout, we give the finite generator set and the corresponding relations of $P_0^{sl(2,\mathbb{C})}$. According to above results, we have a precess of quantum corrections for each generator of $P_0^{sl(2,\mathbb{C})}$, and give the corresponding vertex operators in $S(V_4)^{\Theta_+}$ for each generator of $P_0^{sl(2,\mathbb{C})}$, meanwhile, we also know that $\Gamma : gr(S(V)^{\Theta_+}) \hookrightarrow gr(S(V))^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is a surjection. Finally, by the reconstruction property of $\partial-$ rings, we know $S(V_4)^{\Theta_+}$ is strongly generated by vertex operators obtained above, and give the finite generator set of $S(V_4)^{\Theta_+}$.

Our methods may be effective to lower dimensional highest weight representations V of $sl(2, \mathbb{C})$. More generally, once the dimension of V increases, all three points above mentioned will become much more difficult. So it may be invalid to give a uniform description of commutants of $S(V)$ for higher dimensional representations V by calculating Hilbert series. Therefore, our further work is to seek an more effective way of describing uniformly commutants of $\beta\gamma-$ system $S(V)$ for more general representations V.

2 Lie algebra $sl(2,\mathbb{C})$ and its highest weight module V_4

Let $\{e,f,h\}$ be the standard generators of $sl(2,\mathbb{C})$ satisfying the following relations

$$[e,f]=h,[e,h]=2h,[f,h]=-2f.$$

Its Killing form

$$K(u,v)=tr(adu)(adv),\ \text{for } u,v\in sl(2,\mathbb{C}).$$

It's known that

$$ade=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ -2 & 0 & 0\end{pmatrix}, adh=\begin{pmatrix} 2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 0\end{pmatrix}, adf=\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 2 & 0\end{pmatrix},$$

then there are the relations

$$K(e,e)=K(f,f)=K(e,h)=K(f,h)=0,K(h,h)=8,K(e,f)=4. \tag{2.1}$$

Let x,y be two variables, denote the polynomial algebra with variables x,y by $\mathbb{C}[x,y]$, then a representation $\rho: sl(2,\mathbb{C})\longrightarrow End\mathbb{C}[x,y]$ is given by $\rho(e)=y\frac{\partial}{\partial x},\rho(f)=x\frac{\partial}{\partial y},\rho(h)=y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$. Denote the subspace of homogeneous polynomials with degree 4 of $\mathbb{C}[x,y]$ by V_4. We know that the representation ρ restrict to the subspace V_4, i.e, $\rho: sl(2,\mathbb{C})\longrightarrow End(V_4)$ is an irreducible representation of $sl(2,\mathbb{C})$ with highest weight 4. Obviously, $\{y^4,y^3x,y^2x^2,yx^3,x^4\}$ forms a basis of V_4, denote by $\{\varphi_0,\varphi_1,\cdots,\varphi_4\}$, respectively, where $\varphi_0=y^4$ is a non-zero highest weight vector. Moreover, there are the following relations

$$\rho(e)\varphi_i=i\varphi_{i-1},\rho(f)\varphi_i=(4-i)\varphi_{i+1},\rho(h)\varphi_i=(4-2i)\varphi_i, \tag{2.2}$$

where $i=0,1,\cdots,4$. Let V_4^* be the dual space of V_4, and $\{\xi_0,\xi_1,\cdots,\xi_4\}$ be the dual basis of the dual space V_4^* corresponding to the basis $\{\varphi_0,\varphi_1,\cdots,\varphi_4\}$, then there is an induced representation $\rho^*: sl(2,\mathbb{C})\longrightarrow EndV_4^*$ defined by

$$\rho^*(e)\xi_i=-(i+1)\xi_{i+1},\rho^*(f)\xi_i=-(5-i)\xi_{i-1},\rho^*(h)\xi_i=-(4-2i)\xi_i, \tag{2.3}$$

where $i=0,1,\cdots,4$.

3 Vertex Algebras and Some Examples

In this section, we introduce vertex algebras and give two examples by the points of view in papers [5, 6, 8, 11, 12]. We shall use such vertex algebra notions throughout this paper. The details please refer to [5, 8].

Let V be a vector space over $\mathbb{C}$, and z, w be the formal variables. Denote the space of all linear maps $V \to V((z))$ by $EndV((z))$, where

$$V((z)) := \left\{ \sum_{n\in\mathbb{Z}} v(n)z^{-n-1} \mid v(n) \in V, v(n) = 0 \text{ for } n \gg 0 \right\}.$$

On the space $EndV((z))$, for $n \in \mathbb{Z}$, one can define n-th circle products as follows: For $u, v \in EndV((z))$, the n-th circle products is defined by

$$u(w) \circ_n v(w) = Res_z u(z)v(w)\ell_{|z|>|w|}(z-w)^n - Res_z v(w)u(v)\ell_{|w|>|z|}(z-w)^n.$$

Here $\ell_{|z|>|w|}f(z,w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function $f(z,w)$ in the region $|z| > |w|$.

The non-negative circle products are connected through the operator product expansion (OPE) formula: For $u, v \in EndV((z))$, there are

$$u(z)v(w) = \sum_{n\geq 0} u(w) \circ_n v(w)(z-w)^{-n-1} + : u(z)v(w) :, \tag{3.1}$$

or it is written as $u(z)v(w) \sim \sum\limits_{n\geq 0} u(w) \circ_n v(w)(z-w)^{-n-1}$, where $\sim$ means equal modulo the term $: u(z)v(w) :$. Here, $: u(w)v(w) :$ is a well defined element of $EndV((z))$, called the Wick product of u and v, and there is $: u(w)v(w) := u \circ_{-1} v$. The other circle products are related to this by $n!u(z)_{-n-1}v(z) =: \partial^n u(z)v(z) :$ for non-negative integers n, where ∂ denotes the formal differentiation operator $\frac{d}{dz}$. For $u \in EndV((z))$, there is $1 \circ_n u = \delta_{n,-1}u$ for all $n \in \mathbb{Z}$; $u \circ_n 1 = \delta_{n,-1}u$ for $n \geq -1$.

A linear subspace $U \subset EndV((z))$ containing 1 which is closed under the circle products is called a circle algebra. In particular, U is closed under ∂ since $\partial u = u \circ_{-2} 1$. Let U be a circle algebra, a subset $S = \{u_i | i \in I\}$ of U is said to generate U if any element of U can be written as a linear combination of non-associative words in the letters $u_i, \circ_n$ for $i \in I$ and $n \in \mathbb{Z}$. It is said that S strongly generates U if any element of U can be written as a linear combination of words in the letters $u_i, \circ_n$ for $n < 0$, equivalently, U is spanned by the collection $\{: \partial^{k_1}u_{i_1}(z)\partial^{k_2}u_{i_2}(z)\cdots\partial^{k_m}u_{i_m}(z) : |k_1, k_2, \cdots, k_m \geq 0, m \geq 0\}$.

Definition 3.1. We say that $u, v \in EndV((z))$ circle commute if $(z-w)^N[u(z), v(w)] = 0$ for some $N \geq 0$. Here $[\ ,\ ]$ denotes the bracket. If N can be choose to be zero. we say that u, v commute. A circle algebra is said to be commutative if its elements pairwise circle commute.

The notion of a commutative circle algebra is equivalent to the notion of a vertex algebra. For any commutative circle algebra U, define

$$\begin{aligned}&\pi : U \longrightarrow (EndU)((t))\\ &u \longmapsto \pi(u) : \pi(u)v = \sum_{n\in\mathbb{Z}} (u \circ_n v)t^{-n-1}, \text{ for } \forall v \in U,\end{aligned}$$

then π is an injective circle algebra homomorphism, and the quadruple of structure $(U, \pi, 1, \partial)$ is a vertex algebra. Conversely, if $(V, Y, 1, D)$ is a vertex algebra, the collection $\{Y(v, z)|v \in V\} \subset EndV((z))$ is a commutative circle algebra.

Next, to study the theory of commutants of vertex algebras, we introduce to two associatied examples of vertex algebras.

Example 1(Current algebra). Let $\mathbf{g}$ be a Lie algebra equipped with a symmetric invariant bilinear form B, and $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial algebra over $\mathbb{C}$ with one indeterminate t. The affine Lie algebra $\widehat{\mathbf{g}} = \mathbf{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with bracket

$$[u \otimes t^n, v \otimes t^m] = [u, v] \otimes t^{m+n} + nB(u, v)\delta_{m+n,0}c, \tag{3.2}$$

where c is the center of $\widehat{\mathbf{g}}$.

Set $deg(u\otimes t^n) = n, deg(K) = 0$, then $\widehat{\mathbf{g}}$ is equipped with the $\mathbb{Z}-$ grading structure. Let $\widehat{\mathbf{g}}_{\geq 0} \subset \widehat{\mathbf{g}}$ be the subalgebra of elements of non-negative degree, then $N(\mathbf{g}, B) = \mathfrak{U}(\widehat{\mathbf{g}}) \otimes_{\widehat{\mathbf{g}}_{\geq 0}} \mathbb{C}$ is the induced $\widehat{\mathbf{g}}-$module, where $\mathbb{C}$ is the 1-dimensional $\widehat{\mathbf{g}}_{\geq 0}-$module on which $\mathbf{g} \otimes \mathbb{C}[t]$ acts by zero and c acts by 1. For each $u \in \mathbf{g}$, let $u(n)$ be the linear operator on $N(\mathbf{g}, B)$ representing $u \otimes t^n$, and put $u(z) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-1} \in End(N(\mathbf{g}, B))((z))$. The collection $\{u(z)|u \in \mathbf{g}\}$ generates a vertex algebra in $End(N(\mathbf{g}, B))((z))$, which we denote by $\mathcal{O}(\mathbf{g}, B)$([5, 6, 8, 11]). For any $u, v \in \mathbf{g}$, the vertex operators $u(z), v(z) \in End(N(\mathbf{g}, B))((z))$ satisfy OPE relation

$$u(z)v(w) \sim B(u, v)(z - w)^{-2} + [u, v](w)(z - w)^{-1}. \tag{3.3}$$

Denote $\mathbf{1}$ by the vacuum vector $1 \otimes 1 \in N(\mathbf{g}, B)$, then there is

Lemma 3.2. *(cf.[8]) The creation map $\chi : \mathcal{O}(\mathbf{g}, B) \longrightarrow N(\mathbf{g}, B)$ sending $u(z) \longmapsto u(-1)\mathbf{1}$ is an isomorphism of $\mathcal{O}(\mathbf{g}, B)-$modules.*

If $\mathbf{g}$ is simple, for $\lambda \in \mathbb{C}, \lambda \neq -\frac{1}{2}$, $\mathcal{O}(\mathbf{g}, B)$ has a conformal element $L_{\mathcal{O}}(z) = \frac{1}{2\lambda+1} \sum_{i=1}^{dim\mathbf{g}} : u_i(z)u_i(z) :$, where $\{u_i|i = 1, 2, \cdots, dim\mathbf{g}\}$ is an orthonormal basis of $\mathbf{g}$ with respect to the killing form K.

Example 2($\beta\gamma-$system). Let V be a finite dimensional vector space. Regarding $V \oplus V^*$as an abelian Lie algebra, the affine Lie algebra

$$\eta(V) = (V \oplus V^*) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\tau$$

with bracket

$$[(x,x')\otimes t^n,(y,y')\otimes t^m]=(\langle y',x\rangle-\langle x',y\rangle)\delta_{m+n,0}\tau, \tag{3.4}$$

for $x,y\in V$, and $x',y'\in V^*$. Let $\sigma\subset\eta(V)$ be the subalgebra generated by $\tau,(x,0)\otimes t^n$, and $(0,x')\otimes t^m$ for $n\geq 0, m>0$. Let $\mathbb{C}$ be the 1-dimensional $\sigma-$module on which $(x,0)\otimes t^n$ and $(0,x')\otimes t^m$ act trivially and the central element τ acts by the identity. Denote the linear operators representing $(x,0)\otimes t^n,(0,x')\otimes t^n$ on the induced module $\mathfrak{U}(\eta(V))\otimes_{\mathfrak{U}(\sigma)}\mathbb{C}$ by $\beta^x(n),\gamma^{x'}(n-1)$, respectively, for $n\in\mathbb{Z}$. The power series

$$\beta^x(z)=\sum_{n\in\mathbb{Z}}\beta^x(n)z^{-n-1},\gamma^{x'}(z)=\sum_{n\in\mathbb{Z}}\gamma^x(n)z^{-n-1} \tag{3.5}$$

generate a vertex algebra $S(V)$ in $End(\mathfrak{U}(\eta(V))\otimes_{\mathfrak{U}(\sigma)}\mathbb{C})((z))$, called $\beta\gamma-$system([1]), and the generators satisfy OPE relations

$$\beta^x(z)\gamma^{x'}(w)\sim<x',x>(z-w)^{-1},\beta^x(z)\beta^y(w)\sim 0,\gamma^{x'}(z)\gamma^{y'}(w)\sim 0. \tag{3.6}$$

Suppose that V is a n-dimensional $\mathbf{g}-$module via $\rho:\mathbf{g}\longrightarrow EndV$, where $\mathbf{g}$ is a finite dimensional Lie algebra. There is the following relation between two vertex algebras.

Lemma 3.3. *([8]) The above map ρ induces a vertex algebra homomorphism $\widehat{\rho}:\mathcal{O}(\mathbf{g},B)\longrightarrow S(V)$, where B is the symmetric invariant bilinear form $B(u,v)=-Tr(\rho(u)\rho(v))$ for $u,v\in\mathbf{g}$.*

Here, let $\{x_1,x_2,\cdots,x_n\}$ be a basis of V and $\{x'_1,x'_2,\cdots,x'_n\}$ be a dual basis of V^*, the vertex algebra homomorphism $\widehat{\rho}$ send $u(z)$ to

$$\widehat{u}(z)=-\sum_{i=1}^{n}:\beta^{\rho(u)(x_i)}(z)\gamma^{x'_i}(z):,\forall u\in\mathbf{g}.$$

And there are OPE relations

$$\widehat{u}(z)\widehat{v}(w)\sim B(u,v)(z-w)^{-2}+\widehat{[u,v]}(w)(z-w)^{-1}, \tag{3.7}$$

for $u,v\in\mathbf{g}$. Incidently, $S(V)$ has a conformal element

$$L_S(z)=\sum_{i=1}^{dimV}:\beta^{x_i}(z)\partial\gamma^{x'_i}(z):.$$

Analogous to the commutant construction in the theory of associative algebra, one can has a way to construct subalgebras of a vertex algebra, known as commutant subalgebras.

Definition 3.4. Let W be a vertex algebra and U be any subset of W, the commutant of U in W, denote by $Com(U, W)$, is defined to be the set of vertex operators $v(w) \in W$ commuting strictly with each element of U, that is, $[u(z), v(w)] = 0$ for all $u(z) \in U$.

In terms of above circle product, $[u(z), v(w)] = 0$ if and only if $u(z) \circ_n v(w) = 0$ for all $n \geq 0$, so there is

$$Com(U, W) = \{v(z) \in W | u(z) \circ_n v(z) = 0, \forall u(z) \in U, n \geq 0\} \tag{3.8}$$

Regard W^{U_+} as the subalgebra of invariants in W under the action of U. If Θ is a vertex algebra homomorphic image of a current algebra $\mathcal{O}(\mathbf{g}, B)$, W^{Θ_+} is just the invariant space $W^{\mathbf{g}\otimes\mathbb{C}[t]}$. According to (3.8), the action of Θ on W is induced by the non-negative circle products, hence we can write $Com(\Theta, W)$ as W^{Θ_+}.

According to the vertex algebra homomorphism $\widehat{\rho}$ in Lemma 3.3, there is the following definition

Definition 3.5. Let Θ be the subalgebra $\widehat{\rho}(\mathcal{O}(\mathbf{g}, B)) \subset S(V)$. The commutant algebra $S(V)^{\Theta_+} = Com(\widehat{\rho}(\mathcal{O}(\mathbf{g}, B)), S(V))$ is called the algebra of invariant chiral differential operators on V.

For a simple Lie algebra $\mathbf{g}$ and $B(u, v) = \lambda K(u, v)$ for $\lambda \in \mathbb{C}, \lambda \neq -\frac{1}{2}$, $S(V)^{\Theta_+}$ has the conformal elements $L(z) = L_S(z) - \widehat{\rho}(L_{\mathcal{O}}(z))$. Let $\mathbf{g} = sl(2, \mathbb{C})$, for the representation V_4 of $sl(2, \mathbb{C})$, $B(u, v) = -tr(\rho(u)\rho(v)) = \lambda K(u, v), \forall u, v \in sl(2, \mathbb{C})$. In fact, there is

$$B(e, f) = -20, B(h, h) = -40, B(e, e) = B(f, f) = B(e, h) = B(f, h),$$

then $\lambda = -5$, so there is the vertex algebra homomorphism

$$\widehat{\rho} : \mathcal{O}(sl(2, \mathbb{C}), -5K) \longrightarrow S(V_4).$$

For given basis of V_4 and V_4^* in Section 2, the vertex algebra $S(V_4)$ has a conformal element $L_S(z) = \sum_{i=0}^{4} : \beta^{\varphi_i}(z)\partial\gamma^{\xi_i}(z) :$.

Lemma 3.6. *([8]) The conformal weight-zero subspace* $S(V)_0^{\Theta_+} \subset S(V)^{\Theta_+}$ *coincides with the classical invariant ring* $Sym(V^*)^{\mathbf{g}}$.

Let V be a n-dimensional vector space, $D(V)$ be the Weyl algebra of polynomial differential operators of V, then $D(V)$ has generators $\beta^x, \gamma^{x'}$, which are linear in $x \in V, x' \in V^*$, and satisfies the commutation relations $[\beta^x, \gamma^{x'}] =< x', x >$. If V is a n-dimensional $\mathbf{g}-$module via $\rho : \mathbf{g} \longrightarrow EndV$, there is an induced action ρ^* of $\mathbf{g}$ on $D(V)$, moreover, $\mathbf{g}$ acts on $D(V)$ by derivations of degree 0, and we have $gr(D(V)^{\mathbf{g}}) = gr(D(V))^{\mathbf{g}} = Sym(V \oplus V^*)^{\mathbf{g}}$. The action of $\mathbf{g}$ on $D(V)$ can be realized by inner derivations. In terms of a basis $\{x_1, x_2, \cdots, x_n\}$ of V and the corresponding dual basis $\{x'_1, x'_2, \cdots, x'_n\}$ of V^*, we have a Lie algebra homomorphism $\tau : \mathbf{g} \longrightarrow D(V)$ given by

$$\tau = -\sum_{i=1}^{n} \beta^{\rho(u)(x_i)} \gamma^{x'_i}, \tag{3.9}$$

which can be extended to a map $\mathfrak{U}(\mathbf{g}) \longrightarrow D(V)$, and the action of $u \in \mathbf{g}$ on $v \in D(V)$ is given by $\rho^*(v) = [\tau(u), v]$. Thus $D(V)^{\mathbf{g}} = Com(\tau(\mathfrak{U}(\mathbf{g}), D(V))$.

Let V be a n-dimensional $\mathbf{g}$- module equipped with a symmetric invariant bilinear form B'. If $\{x_1, x_2, \cdots, x_n\}$ is an orthonormal basis of V with respect to B', and $\{x'_1, x'_2, \cdots, x'_n\}$ is the corresponding dual basis of V^*. there are the following results(cf. [8, 11])

Lemma 3.7. *There is a Lie algebra homomorphism $\psi : sl(2, \mathbb{C}) \longrightarrow D(V)^{\mathbf{g}}$ given in an orthonormal basis with respect to B' by the formulas*

$$h \longmapsto \sum_{i=1}^{n} \beta^{x_i} \gamma^{x'_i}, e \longmapsto \frac{1}{2} \sum_{i=1}^{n} \gamma^{x'_i} \gamma^{x'_i}, f \longmapsto -\frac{1}{2} \sum_{i=1}^{n} \beta^{x_i} \beta^{x_i}. \tag{3.10}$$

Lemma 3.8. *The homomorphism $\psi : sl(2, \mathbb{C}) \longrightarrow D(V)^{\mathbf{g}}$ induces a vertex algebra homomorphism $\widehat{\psi} : \mathcal{O}(sl(2, \mathbb{C}), -\frac{n}{8}K) \longrightarrow S(V)^{\Theta_+}$ by*

$$h(z) \longmapsto v^h(z) = \sum_{i=1}^{n} : \beta^{x_i}(z) \gamma^{x'_i}(z) :, \tag{3.11}$$

$$e(z) \longmapsto v^e(z) = \frac{1}{2} \sum_{i=1}^{n} : \gamma^{x'_i}(z) \gamma^{x'_i}(z) :, \tag{3.12}$$

$$f(z) \longmapsto v^f(z) = -\frac{1}{2} \sum_{i=1}^{n} : \beta^{x_i}(z) \beta^{x_i}(z) :, \tag{3.13}$$

where K is the Killing form of $sl(2, \mathbb{C})$.

Since V_4 is isomorphic to V_4^* as $sl(2, \mathbb{C})-$ modules, such an isomorphism can determine a symmetric invariant bilinear form B'. Here we define a linear map

$$\begin{aligned} \varphi : V_4 &\longrightarrow V_4^* \\ \varphi_0 &\longmapsto \xi_4, \\ \varphi_1 &\longmapsto -\tfrac{1}{4}\xi_3, \\ \varphi_2 &\longmapsto \tfrac{1}{6}\xi_2, \\ \varphi_3 &\longmapsto -\tfrac{1}{4}\xi_1, \\ \varphi_4 &\longmapsto \xi_0. \end{aligned}$$

It is easy to check φ is an $sl(2, \mathbb{C})-$ module isomorphism, then we define a bilinear form $B'(\cdot, \cdot)$ on V_4 as follows $B'(\varphi_i, \varphi_j) = \langle \varphi(\varphi_i), \varphi_j \rangle$, where $\langle , \rangle$ is the pair of V_4 and V_4^*. Let $k_0 = 1, k_1 = \frac{1}{4}, k_2 = \frac{1}{6}, k_3 = \frac{1}{4}, k_4 = 1$, we have

$$B'(\varphi_i, \varphi_j) = \langle (-1)^i k_i \xi_{4-i}, \varphi_j \rangle = \begin{cases} (-1)^i k_i, & 4 - i = j, \\ 0, & 4 - i \neq j, \end{cases}$$

and

$$B'(\varphi_j,\varphi_i)=\langle(-1)^j k_j\xi_{4-j},\varphi_i\rangle=\begin{cases}(-1)^i k_i, & 4-j=i,\\ 0, & 4-j\neq i,\end{cases}$$

Hence $B'(\varphi_i,\varphi_j)=B'(\varphi_j,\varphi_i)$, and since $B'(\cdot,\cdot)$ is bilinear, so it is symmetric.

On the other hand, there are

$$\begin{aligned}&B'(\rho(e)(\varphi_i),\varphi_j)+B'(\varphi_i,\rho(e)(\varphi_j))\\&=\begin{cases}i(-1)^{i-1}k_{i-1}+(4-i+1)(-1)^i k_i, & 4-i+1=j,\\ 0, & 4-i+1\neq j,\end{cases}\\&=0.\end{aligned}$$

$$\begin{aligned}&B'(\rho(h)(\varphi_i),\varphi_j)+B'(\varphi_i,\rho(h)(\varphi_j))\\&=\begin{cases}(4-2i)(-1)^i k_i+(4-2j)(-1)^i k_i, & 4-i=j,\\ 0, & 4-i\neq j,\end{cases}\\&=0.\end{aligned}$$

$$\begin{aligned}&B'(\rho(f)(\varphi_i),\varphi_j)+B'(\varphi_i,\rho(f)(\varphi_j))\\&=\begin{cases}(4-i)(-1)^{i+1}k_{i+1}+(4-j)(-1)^i k_i, & 4-i=j+1,\\ 0, & 4-i\neq j+1,\end{cases}\\&=0.\end{aligned}$$

Since $B'(\cdot,\cdot)$ is bilinear, so it is $sl(2,\mathbb{C})-$invariant on V_4. Moreover, we know

$$B'(\varphi_i,\varphi_j)=\begin{cases}1, & i=0,j=4,\\ -\frac{1}{4}, & i=1,j=3,\\ \frac{1}{6}, & i=2,j=2,\\ 0, & i\neq 4-j.\end{cases}$$

Hence we can determine an orthonormal basis of V_4 with respect to $B'(\cdot,\cdot)$ as follows

$$\left\{\frac{1}{\sqrt{2}}(\varphi_0+\varphi_4),\sqrt{-2}(\varphi_1+\varphi_3),\sqrt{6}\varphi_2,\frac{1}{\sqrt{-2}}(\varphi_0-\varphi_4),\frac{1}{\sqrt{2}}(\varphi_1-\varphi_3)\right\},$$

denote by $\{x_0,x_1,\cdots,x_4\}$. The corresponding dual basis of V_4 is

$$\left\{\frac{1}{\sqrt{2}}(\xi_0+\xi_4),\frac{1}{2\sqrt{-2}}(\xi_1+\xi_2),\frac{1}{\sqrt{6}}\xi_2,\frac{1}{\sqrt{-2}}(\xi_0-\xi_4),\frac{1}{2\sqrt{2}}(\xi_1-\xi_4)\right\},$$

denote by $\{x'_0,x'_1,\cdots,x'_4\}$. By Lemma 3.8, there is

Corollary 3.9. *For the representation V_4, ψ can induce a vertex algebra homomorphism $\widehat{\psi}:\mathcal{O}(sl(2,\mathbb{C}),-\frac{5}{8}K)\longrightarrow S(V_4)^{\Theta_+}$ by*

$$h(z)\longmapsto v^h(z)=\sum_{i=0}^{4}:\beta^{\varphi_i}(z)\gamma^{\xi_i}(z):,\tag{3.14}$$

$$e(z) \longmapsto v^e(z) =: \gamma^{\xi_0}(z)\gamma^{\xi_4}(z) : -\frac{1}{4} : \gamma^{\xi_1}(z)\gamma^{\xi_3}(z) : +\frac{1}{12} : \gamma^{\xi_2}(z)\gamma^{\xi_2}(z) :, \tag{3.15}$$

$$f(z) \longmapsto v^f(z) = - : \beta^{\varphi_0}(z)\beta^{\varphi_4}(z) : +4 : \beta^{\varphi_1}(z)\beta^{\varphi_3}(z) : -3 : \beta^{\varphi_2}(z)\beta^{\varphi_2}(z) : . \tag{3.16}$$

4 The Category $\mathfrak{R}$ and Category of $\partial-$ Rings

In this section, we introduce two categories and their properties. Refer to [8, 9, 11, 12] for details.

Let $\mathfrak{R}$ be the category of pairs (W, deg), where W is a vertex algebra equipped with a $\mathbb{Z}_{\geq 0}-$filtration

$$W_0 \subset W_1 \subset W_2 \subset \cdots, W = \bigcup_{k \geq 0} W_k,$$

such that $W_0 = \mathbb{C}$, and for $a \in W_k, b \in W_l$, there are

$$a \circ_n b \in W_{k+l}, \text{ for } n < 0, \tag{4.1}$$

$$a \circ_n b \in W_{k+l-1}, \text{ for } n \geq 0. \tag{4.2}$$

Here $W_k = 0$ for $k < 0$. A non-zero element $a(z) \in W$ is said to have degree d if d is the minimal integer for which $a(z) \in W_d$. Morphisms in $\mathfrak{R}$ are morphisms of vertex algebras which preserve the above filtration. Filtration on vertex algebras satisfying (4.1),(4.2) are known as good increasing filtration ([9]) . If W possesses such a filtration, it follows that the associated graded object $gr(W) = \bigoplus_{k>0} W_k/W_{k-1}$ is a $\mathbb{Z}_{\geq 0}-$ graded associative, commutative algebra with a unit 1 under a product induced by the Wick product on W. Moreover, $gr(W)$ has a derivation ∂ of degree zero 0, and for each $a \in W_d$ and $n \geq 0$, operators $a \circ_n$ on W induces a derivation of degree $d-1$ on $gr(W)$. For each $d \geq 1$, we have the projection $\phi_d : W_d \longrightarrow W_d/W_{d-1} \in gr(W)$.

If $u, v \in gr(W)$ are homogeneous of degree r, s, respectively, and $u(z) \in W_r, v(z) \in W_s$ are vertex operators such that $\phi_r(u(z)) = u$ and $\phi_s(v(z)) = v$, it follows that $\phi_{r+s}(: u(z)v(z) :) = uv$.

Let $\mathcal{R}$ denote the category of $\mathbb{Z}_{\geq 0}-$graded commutative rings equipped with a derivation ∂ of degree zero, which is called $\partial-$rings.

The prominent feature of $\mathfrak{R}$ is that vertex algebra $W \in \mathfrak{R}$ has the following reconstruction property. We can write down a set of strong generators for W as a vertex algebra just by studying the ring structure of $gr(W)$. We say that the collection $\{a_i | i \in I\}$ generates $gr(W)$ as a $\partial-$ring if the collection $\{\partial^k a_i | i \in I, k \geq 0\}$ generates $gr(W)$ as a grading ring.

Lemma 4.1. *Let W be a vertex algebra in $\mathfrak{R}$. Suppose that $gr(W)$ is generated as a $\partial-$ ring by a collection $\{a_i | i \in I\}$, where a_i is homogeneous of degree d_i, choose vertex operators $a_i(z) \in W_{d_i}$ such that $\phi_{d_i}(a_i(z)) = a_i$, then W is strongly generated by the collection $\{a_i(z) | i \in I\}$.*

Define $k = k(W, deg) = Sup\{j \geq 1 | W_r \circ_n W_s \subset W_{r+s-j}, \forall r, s, n \geq 0\}$, it follows easily that k is finite if and only if W is not abelian(cf. [8]). For two vertex algebras $\mathcal{O}(\mathbf{g}, B)$ and $S(V)$, there are $k(\mathcal{O}(\mathbf{g}, B), deg) = 1, k(S(V), deg) = 2$.

Lemma 4.2. *([8]) Let $(W, deg) \in \mathfrak{R}$, and $k = k(W, deg)$ be as above. For each $u(z) \in W$ of degree d and $n \geq 0$, the operator $u(z)\circ_n$ on W induces a homogeneous derivation $u(n)_{Der}$ on $gr(W)$ of degree $d - k$, defined on homogeneous elements v of degree r by*

$$u(n)_{Der}(v) = \phi_{r+d-k}(u(z) \circ_n v(z)). \tag{4.3}$$

Here $v(z) \in W$ is any vertex operator of degree r such that $\phi_r(v(z)) = v$.

Let $(W, deg) \in \mathfrak{R}, \mathcal{C}$ be a subalgebra of W which is a homomorphic image of a current algebra $\mathcal{O}(\mathbf{g}, B)$. Suppose that for each $u \in \mathbf{g}$, $u(z) \in \mathcal{C}$ has degree k, so that the derivations $\{u(n)_{Der}|n \geq 0\}$ on $gr(W)$ are homogeneous of degree 0. Then there is

Lemma 4.3. *([8]) The derivations $\{u(n)_{Der}|n \geq 0\}$ form a representation of $\mathbf{g} \otimes \mathbb{C}[t]$ on $gr(W)$. Moreover, the actions of $\mathbf{g} \otimes \mathbb{C}[t]$ on W and $gr(W)$ are compatible in the sense that for any $w(z) \in W$ of degree r, there are $u(n)_{Der}\phi_r(w(z)) = \phi_r \circ u(n)(w(z))$.*

Consider the invariant ring $gr(W)^{\mathcal{C}_+}$, since $gr(W)^{\mathcal{C}_+}$ is closed under ∂, then $W^{\mathcal{C}_+} \hookrightarrow W$ induces an injective ring homomorphism $gr(W^{\mathcal{C}_+}) \hookrightarrow gr(W)$ whose image clearly lies in $gr(W)^{\mathcal{C}_+}$. So we have a canonical injective homomorphism $\Gamma : gr(W^{\mathcal{C}_+}) \hookrightarrow gr(W)^{\mathcal{C}_+}$ as $\partial-$rings. Let $W = S(V), \Theta = \widehat{\rho}(\mathcal{O}(\mathbf{g}, B))$, where $\mathbf{g}$ is semisimple and V is a finite dimensional $\mathbf{g}$-module. In the case, $deg(\widehat{u}(z)) = 2 = k$, so each $\widehat{u}(n)_{Der}$ is homogeneous of degree 0 and $gr(S(V)$ is a $\mathbf{g} \otimes \mathbb{C}[t]-$module by Lemma 4.3. Denote by $P = gr(S(V))$, and we write the images of $\partial^k\beta^x(z), \partial^k\gamma^{x'}(z)$ in P as β_k^x and $\gamma_k^{x'}$, respectively. The action of $\widehat{u}(n)_{Der}$ on the generators of P is given by

$$\widehat{u}(n)_{Der}(\beta_k^x) = C_k^n\beta_{k-n}^{\rho(u)(x)}; \widehat{u}(n)_{Der}(\gamma_k^{x'}) = C_{k-n}^n\gamma^{\rho^*(u)(x')}, \tag{4.4}$$

where $C_k^n = k(k-1)\cdots(k-n+1)$ for $n, k \geq 0$, $C_k^0 = 1, C_k^n = 0$ for $n > k$.

If V admits a symmetric, $\mathbf{g}$-invariant bilinear form, so $S(V)^{\Theta_+}$ contains the subalgebra $\mathcal{A} = \widehat{\psi}(\mathcal{O}(sl(2,\mathbb{C}), -\frac{dimV}{8}K))$, the operators $\{v^u(n)_{Der}|u \in sl(2,\mathbb{C}), n \geq 0\}$ on P form a representation of the Lie algebra $sl(2,\mathbb{C}) \otimes \mathbb{C}[t]$ by derivations of degree 0. In terms of an orthonormal basis of V, the action is as follows

$$v^h(n)_{Der}(\beta_k^{x_i}) = -C_k^n\beta_{k-n}^{x_i}; v^h(n)_{Der}(\gamma_k^{x_i'}) = C_k^n\gamma_{k-n}^{x_i'} \tag{4.5}$$

$$v^e(n)_{Der}(\beta_k^{x_i}) = -\frac{1}{2}C_k^n\gamma_{k-n}^{x_i'}; v^e(n)_{Der}(\gamma_k^{x_i'}) = 0; \tag{4.6}$$

$$v^f(n)_{Der}(\beta_k^{x_i}) = 0; v^f(n)_{Der}(\gamma_k^{x_i'}) = \frac{1}{2} C_k^n \beta_{k-n}^{x_i} \tag{4.7}$$

Next, we shall state a conclusion in [12], which plays an important role for this paper.

Let V be a linear representation of G (connected, reductive linear algebraic group over $\mathbb{C}$ with Lie algebra $\mathbf{g}$). Choose a basis $\{x_1, x_2, \cdots, x_n\}$ for V^*, so the regular function ring $\mathcal{O}(V) \cong \mathbb{C}[x_1, x_2, \cdots, x_n]$, and there is $P = gr(S(V)) = \mathbb{C}[\beta_k^x, \gamma_k^{x'} | x \in V, x' \in V^*, k \geq 0]$.

Lemma 4.4. *([12]) Let G be a connected, reductive algebraic group, and let V be a linear $G-$ representation such that $\mathcal{O}(V \oplus V^*)$ contains no invariant lines, then $P^{\mathbf{g}\otimes\mathbb{C}[t]}$ is generated by*

$$P_0^G = \mathbb{C}[\beta_0^x, \gamma_0^{x'}]^G \cong \mathcal{O}(V \oplus V^*)^G$$

as a $\partial-$ ring. In particular, if $\{f_1, f_2, \cdots, f_n\}$ generate P_0^G as a ring, then $\{f_1, f_2, \cdots, f_n\}$ generate $P^{\mathbf{g}\otimes\mathbb{C}[t]}$ as a $\partial-$ ring.

At the same times, we have the well-known theorem(cf.[13])

Lemma 4.5. *(Hilbert Theorem) If G is a connected, reductive algebraic group, then the invariant ring of polynomials $\mathbb{C}[x_1, x_2, \cdots, x_n]^G$ is finitely generated.*

For Lie algebra $sl(2, \mathbb{C})$ and the representation V_4, the ring $P = gr(S(V_4)) = \mathbb{C}[\beta_k^{\varphi_i}, \gamma_k^{\xi_i} | i = 0, 1, \cdots, 4, k \geq 0]$ and $P_0 = \mathbb{C}[\beta_k^{\varphi_i}, \gamma_k^{\xi_i} | i = 0, 1, \cdots, 4]$. Since special linear group $SL(2)$ is a connected, linear reductive algebraic group with the Lie algebra $sl(2, \mathbb{C})$. By Lemma 4.4, $P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by $P_0^{sl(2,\mathbb{C})} = \mathbb{C}[\beta_0^{\varphi_i}, \gamma_0^{\xi_i} | i = 0, 1, \cdots, 4]^{sl(2,\mathbb{C})}$ as a $\partial-$ ring. By Lemma 4.5, the invariant ring $P_0^{sl(2,\mathbb{C})}$ is finitely generated, so $P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is finitely generated as $\partial-$ ring. Next, we would like to describe the ring $P_0^{sl(2,\mathbb{C})}$ by giving its generators.

5 Hilbert series and Generators of $P_0^{sl(2,\mathbb{C})}$

In this section, we shall calculate the Hilbert series of $V_4 \oplus V_4^*$, then give the generators of $P_0^{sl(2,\mathbb{C})}$, which are also the generators of P^{Θ_+} as $\partial-$ ring. Here, we refer to the related definitions and results in [13].

Definition 5.1. Let G be a subgroup of general linear group $GL(n)$, $T = \mathbb{C}[x_1, x_2, \cdots, x_n]$ be the polynomial ring, G has an action on T, denoted by T^G for the ring of invariants of G. $T = \bigoplus_{d\geq 0} T_d$, where $T_d \subset T$ is the subspace of homogeneous polynomials of degree d, then $T^G = \bigoplus_{d\geq 0} T^G \cap T_d$. There is a formal power series in an indeterminate t

$$P(t) = \sum_{d\geq 0} dim(T^G \cap T_d)t^d \in \mathbb{Z}[[t]] \tag{5.1}$$

is called the Hilbert series of the grading ring T^G.

In terms of the Hilbert series, there are the two important theorems(cf.[13])

Lemma 5.2. *([13]) If T^G is generated by homogeneous polynomials $f_1, f_2, \cdots, f_r$ of degree $d_1, d_2, \cdots, d_r$, then the Hilbert series of T^G is the power series expansion at $t = 0$ of rational function*

$$P(t) = \frac{F(t)}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_r})} \tag{5.2}$$

for some $F(t) \in \mathbb{Z}[t]$.

Let V be any $n-$ dimensional representation of $sl(2,\mathbb{C})$, consider the induced action of $sl(2,\mathbb{C})$ on the polynomial ring $T(V) = \mathbb{C}[x_1, x_2, \cdots, x_n]$ of functions on V. Let $a_1, a_2, \cdots, a_n \in \mathbb{Z}$ be the weights (not necessarily distinct) of $sl(2,\mathbb{C})$ in the weight-space decomposition of V, the function

$$P(q;t) = \frac{1}{(1-q^{a_1}t)(1-q^{a_2}t)\cdots(1-q^{a_n}t)} = det\left(I_V - t\left(\begin{array}{cc} q & 0 \\ 0 & q^{-1} \end{array}\right)_V\right)^{-1} \tag{5.3}$$

is called the $q-$Hilbert series of the representation V. There is the relations between $q-$Hilbert series and Hilbert series as follows

Proposition 5.3. *([13]) The invariant ring $T(V)^{sl(2,\mathbb{C})}$ has Hilbert series*

$$P(t) = Res_{q=0}(q - q^{-1})P(q;t). \tag{5.4}$$

Equivalently, if $P(q;t) = \sum_{m\in\mathbb{Z}} c_{(m)}(t)q^m$, then $P(t) = c_{(0)}(t) - c_{(-2)}(t)$.

For the irreducible representation V_4, we can know that $(S(V_4), L(z))$ forms a conformal vertex algebra. By Lemma 3.6, we know that the weight-zero subspace $S(V_4)_0^{\Theta_+}$ coincides with the classical invariant ring $Sym(V_4^*)^{sl(2,\mathbb{C})}$. Since $Sym(V_4^*)^{sl(2,\mathbb{C})} = \mathbb{C}[\gamma^{\xi_0}, \gamma^{\xi_1}, \cdots, \gamma^{\xi_4}]^{sl(2,\mathbb{C})}$, there is the following proposition

Proposition 5.4. *The invariant ring $Sym(V_4^*)^{sl(2,\mathbb{C})}$ is generated by two elements $g_2(\xi), g_3(\xi)$, where*

$$g_2(\xi) = \gamma^{\xi_0}\gamma^{\xi_4} - \frac{1}{4}\gamma^{\xi_1}\gamma^{\xi_3} + \frac{1}{12}\gamma^{\xi_2}\gamma^{\xi_2}, \tag{5.5}$$

$$g_3(\xi) = 8\gamma^{\xi_0}\gamma^{\xi_2}\gamma^{\xi_4} - 3\gamma^{\xi_0}\gamma^{\xi_3}\gamma^{\xi_3} - 3\gamma^{\xi_1}\gamma^{\xi_1}\gamma^{\xi_4} + \gamma^{\xi_1}\gamma^{\xi_2}\gamma^{\xi_3} - \frac{2}{9}\gamma^{\xi_2}\gamma^{\xi_2}\gamma^{\xi_2}. \tag{5.6}$$

Proof. For V_4, we know that Hilbert series is $P(t)=\frac{1}{(1-t^2)(1-t^3)}$(cf.[13]), hence the invariant ring has two generators: one has degree 2, another has degree 3. Using the action relations (2.3) and (4.4), we can check $g_2(\xi), g_3(\xi) \in \mathbb{C}[\gamma^{\xi_0}, \cdots, \gamma^{\xi_4}]^{sl(2,\mathbb{C})}$. Obviously, $g_2(\xi), g_3(\xi)$ are algebraically independent, so $\mathbb{C}[\gamma^{\xi_0}, \cdots, \gamma^{\xi_4}]^{sl(2,\mathbb{C})}$ is generated by $g_2(\xi), g_3(\xi)$.

Definition 5.5. The $q-$analogue of an integer d, its factorial and binomial coefficients are

$$(1)\quad [d]_q = q^{d-1} + q^{d-3} + \cdots + q^{-d+3} + q^{-d+1} = \frac{q^d - q^{-d}}{q - q^{-1}},$$

$$(2)\quad [d]_q! = \prod_{i=1}^{d} [i]_q;\ (3)\quad \left[\begin{array}{c} d+r \\ r \end{array}\right]_q = \frac{[d+r]_q!}{[d]_q![r]_q!}.$$

Analogues to the classical binomial theorem $\frac{1}{(1-t)^{d+1}} = \sum_{r\geq 0} \left(\begin{array}{c} d+r \\ r \end{array}\right) t^r$, there is

Proposition 5.6. *([13])*

$$\prod_{i=0}^{d} \frac{1}{1-q^{d-2i}t} = \sum_{r\geq 0} \left[\begin{array}{c} d+r \\ r \end{array}\right]_q t^r.$$

Lemma 5.7. *([13]) The Hilbert series of the invariant ring* $\mathbb{C}[\gamma^{\xi_0}, \cdots, \gamma^{\xi_d}]^{sl(2,\mathbb{C})}$ *for binary forms of degree d is given by*

$$P(t) = -\sum_{r\geq 0} \left\{ Res_{q=0}(q-q^{-1}) \left[\begin{array}{c} d+r \\ r \end{array}\right]_q \right\} t^r. \tag{5.7}$$

For convenience, denote by $u = q^2$, let $f(u) \in \mathbb{Z}[[u]]$, denote the coefficient of u^n by $[f(u)]_n \in \mathbb{Z}$.

Caykey – Sylvester Formula(cf. [13]) The vector space $\mathbb{C}[\gamma^{\xi_0}, \cdots, \gamma^{\xi_d}]_r^{sl(2,\mathbb{C})}$ of invariants of degree r has dimension

$$m(d,r) = \begin{cases} \left[\frac{(1-u^{r+1})(1-u^{r+2})\cdots(1-u^{r+d})}{(1-u)(1-u^2)\cdots(1-u^d)}\right]_{\frac{dr}{2}}, & dr \text{ is even}, \\ 0, & dr \text{ is odd}. \end{cases}$$

For the representation $V_4 \oplus V_4^*$, we have known that there are basis $\{\varphi_0, \cdots, \varphi_4, \xi_0, \cdots, \xi_4\}$, its $q-$Hilbert series is therefore

$$P(q,t) = \prod_{i=0}^{4} \frac{1}{(1-q^{4-2i}t)^2}. \tag{5.8}$$

If we know the Hilbert series of $V_4 \oplus V_4^*$, we can describe the invariant ring $P_0^{sl(2,\mathbb{C})}$ and its generators, where $P_0 = \mathbb{C}[\beta_0^{\varphi_0}, \cdots, \beta_0^{\varphi_4}; \gamma_0^{\xi_0}, \cdots, \gamma_0^{\xi_4}]$.

Proposition 5.8. *For the representation $V_4 \oplus V_4^*$ of $sl(2,\mathbb{C})$, the corresponding Hilbert series is*

$$P(t) = \frac{1-t^{12}}{(1-t^2)^3(1-t^3)^4(1-t^4)}. \tag{5.9}$$

Proof. Firstly, we need to expand the $q-$Hilbert series as form of $\sum\limits_{m\in\mathbb{Z}} C_{(m)}(t)q^m$.

$$\begin{aligned} P(q,t) &= \prod_{i=0}^{4} \frac{1}{(1-q^{4-2i}t)^2} \\ &= \frac{1}{(1-t)^2}\left(\frac{1}{(1-q^4t)^2(1-q^2t)^2(1-q^{-2}t)^2(1-q^{-4}t)^2}\right) \\ &= \frac{1}{(1-t)^2}\left(\frac{q^{12}}{(1-q^4t)^2(1-q^2t)^2(q^2-t)^2(q^4-t)^2}\right) \end{aligned}$$

Denote by $\overline{P(q,t)} = \frac{q^{12}}{(1-q^4t)^2(1-q^2t)^2(q^2-t)^2(q^4-t)^2}$, we suppose that

$$\begin{aligned} \overline{P(q,t)} &= \frac{q^{12}}{(1-q^4t)^2(1-q^2t)^2(q^2-t)^2(q^4-t)^2} \\ &= \frac{a_1q^3+b_1q^2+c_1q+d_1}{1-q^4t} + \frac{a_2q^3+b_2q^2+c_2q+d_2}{(1-q^4t)^2} \\ &+ \frac{a_3q+b_3}{1-q^2t} + \frac{a_4q+b_4}{(1-q^2t)^2} + \frac{a_5q+b_5}{q^2-t} + \frac{a_6q+b_6}{(q^2-t)^2} \\ &+ \frac{a_7q^3+b_7q^2+c_7q+d_7}{q^4-t} + \frac{a_8q^3+b_8q^2+c_8q+d_8}{(q^4-t)^2} \end{aligned}$$

Where $a_1,\cdots,a_8,b_1,\cdots,b_8,c_1,c_2,c_7,c_8,d_1,d_2,d_7,d_8$ are all undetermined coefficients with indeterminate t. Using the methods of undetermined coefficients, we solve that

$$
\begin{aligned}
&a_1 = a_2 = \cdots = a_8 = 0, c_1 = c_2 = c_7 = c_8 = 0,\\
&b_1 = \frac{-2t^2(t^2-t-1)(t^2+1)}{(t-1)^7(t+1)(t^2+t+1)^3}, b_2 = \frac{2t(t^2+1)}{(t-1)^6(t+1)(t^2+t+1)^2},\\
&b_3 = \frac{-2t^2(3t^4+5t^3+5t^2+4t+2)}{(t-1)^7(t+1)^3(t^2+t+1)^3}, b_4 = \frac{t^2}{(t+1)^2(t^2+t+1)^2(t-1)^6},\\
&b_5 = \frac{-2t^3(2t^4+4t^3+5t^2+5t+3)}{(t-1)^7(t+1)^3(t^2+t+1)^3}, b_6 = \frac{t^4}{(t-1)^6(t^2+t+1)^2(t+1)^2},\\
&b_7 = \frac{2t(t^2+1)(t^2+t-1)}{(t-1)^7(t+1)(t^2+t+1)^3}, b_8 = \frac{2(t^2+1)t^2}{(t-1)^6(t+1)(t^2+t+1)^2},\\
&d_1 = \frac{-t(2t^7+t^6+5t^5-t^4-6t^3-9t^2-5t-3)}{(t-1)^7(t+1)^3(t^2+t+1)^3},\\
&d_2 = \frac{t^4+t^3+4t^2+t+1}{(t-1)^6(t+1)^2(t^2+t+1)^2}, d_7 = \frac{t(3t^7+5t^6+9t^5+6t^4+t^3-5t^2-t-2)}{(t-1)^7(t+1)^3(t^2+t+1)^3},\\
&d_8 = \frac{t^2(t^4+t^3+4t^2+t+1)}{(t-1)^6(t+1)^2(t^2+t+1)^2}.
\end{aligned}
$$

Using the following expansions of rational functions

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n; \frac{1}{1+u} = \sum_{n=0}^{\infty}(-1)^n u^n; \frac{1}{(1-u)^2} = \sum_{n=0}^{\infty}(n+1)u^n,$$

we get the following expansion

$$
\begin{aligned}
\overline{P(q,t)} &= \frac{b_1q^2+d_1}{1-q^4t} + \frac{b_2q^2+d_2}{(1-q^4t)^2} + \frac{b_3}{1-q^2t} + \frac{b_4}{(1-q^2t)^2}\\
&\quad + \frac{b_5}{q^2-t} + \frac{b_6}{(q^2-t)^2} + \frac{b_7q^2+d_7}{q^4-t} + \frac{b_8q^2+d_8}{(q^4-t)^2}\\
&= (b_1q^2+d_1)\sum_{k=0}^{+\infty} t^k q^{4k} + (b_2q^2+d_2)\sum_{k=1}^{+\infty} kt^{k-1}q^{4(k-1)} + b_3\sum_{k=0}^{+\infty} t^k q^{2k}\\
&\quad + b_4\sum_{k=1}^{+\infty} kt^{k-1}q^{2(k-1)} + b_5\sum_{k=0}^{+\infty} t^k q^{-2k-2} + b_6\sum_{k=1}^{+\infty} kt^{k-1}q^{-2k-2}\\
&\quad + (b_7q^2+d_7)\sum_{k=0}^{+\infty} t^k q^{-4k-4} + (b_8q^2+d_8)\sum_{k=1}^{+\infty} kt^{k-1}q^{-4k-4}\\
&= \sum_{m\in\mathbb{Z}} \overline{C}_{(m)}(t)q^m.
\end{aligned}
$$

hence we obtain

$$\overline{C}_{(0)}(t) = d_1 + d_2 + b_3 + b_4; \overline{C}_{(-2)}(t) = b_5 + b_7.$$

By Proposition 5.3, we can compute the Hilbert series

$$
\begin{aligned}
P(t) &= \frac{1}{(1-t)^2}(\overline{C}_{(0)}(t) - \overline{C}_{(-2)}(t)) = \frac{1}{(1-t)^2}(d_1 + d_2 + b_3 + b_4 - b_5 - b_7) \\
&= \frac{-(t^2-t+1)(t^4-t^2+1)}{(t-1)^7(t+1)^3(t^2+t+1)^3} = \frac{(t^2-t+1)(t^4-t^2+1)}{(1-t^3)^3(1-t^2)^3(1-t)} \\
&= \frac{(t^2+t+1)(t^2-t+1)(t^4-t^2+1)}{(1-t^3)^3(1-t^2)^3(1-t)(t^2+t+1)} = \frac{(t^4-t^2+1)(t^4+t^2+1)}{(1-t^3)^4(1-t^2)^3} \\
&= \frac{t^8+t^4+1}{(1-t^3)^4(1-t^2)^3} = \frac{(t^8+t^4+1)(1-t^4)}{(1-t^3)^4(1-t^2)^3(1-t^4)} \\
&= \frac{1-t^{12}}{(1-t^3)^4(1-t^2)^3(1-t^4)}.
\end{aligned}
$$

Remark 5.9. The above proposition shows some information about generators of the invariant ring $P_0^{sl(2,\mathbb{C})} = \mathbb{C}[\beta_0^{\varphi_0}, \cdots, \beta_0^{\varphi_4}; \gamma_0^{\xi_0}, \cdots, \gamma_0^{\xi_4}]^{sl(2,\mathbb{C})}$: It has eight generators, three of them have degree 2, four of them have degree 3, and one of them has degree 4. Meanwhile, such eight generators satisfy a polynomial relation of degree 12.

Next we consider dimensions of some homogeneous subspaces of the invariant ring $P_0^{sl(2,\mathbb{C})}$.

We consider the representation $V_4 \oplus V_4^*$, analogue to Proposition 5.6, Lemma 5.7 , there are the following results about the $q-$ Hilbert series and Hilbert series

Proposition 5.10.

$$
\begin{aligned}
P(q,t) &= \prod_{i=0}^{4} \tfrac{1}{(1-q^{4-2i}t)^2} \\
&= \sum_{k=0}^{+\infty} \left(\sum_{r=0}^{k} \begin{bmatrix} 4+r \\ r \end{bmatrix}_q \begin{bmatrix} 4+k-r \\ k-r \end{bmatrix}_q \right) t^k.
\end{aligned}
$$

Proposition 5.11. *The Hilbert series of invariant ring $P_0^{sl(2,\mathbb{C})}$ is given by*

$$
P(t) = -\sum_{k=0}^{+\infty} \left\{ Res_{q=0}(q-q^{-1}) \left(\sum_{r=0}^{k} \begin{bmatrix} 4+r \\ r \end{bmatrix}_q \begin{bmatrix} 4+k-r \\ k-r \end{bmatrix}_q \right) \right\} t^k.
$$

Let $u = q^2$, since there are

$$
\begin{bmatrix} 4+r \\ r \end{bmatrix}_q = \tfrac{[r+1]_q \cdots [r+4]_q}{[1]_q \cdots [4]_q} = q^{-4r} \tfrac{(1-u^{r+1})\cdots(1-u^{r+4})}{(1-u)(1-u^2)\cdots(1-u^4)},
$$

$$
\begin{bmatrix} 4+k-r \\ k-r \end{bmatrix}_q = \tfrac{[k-r+1]_q \cdots [k-r+4]_q}{[1]_q \cdots [4]_q} = q^{-4(k-r)} \tfrac{(1-u^{k-r+1})\cdots(1-u^{k-r+4})}{(1-u)(1-u^2)\cdots(1-u^4)},
$$

then

$$
\begin{aligned}
&-(q-q^{-1})\left(\sum_{r=0}^{k}\left[\begin{array}{c}4+r\\ r\end{array}\right]_q\left[\begin{array}{c}4+k-r\\ k-r\end{array}\right]_q\right).\\
&=\sum_{r=0}^{k}-(q-q^{-1})q^{-4k}\tfrac{(1-u^{r+1})\cdots(1-u^{r+4})(1-u^{k-r+1})\cdots(1-u^{k-r+4})}{(1-u)^2(1-u^2)^2\cdots(1-u^4)^2}\\
&=\sum_{r=0}^{k}q^{-4k-1}\tfrac{(1-u^{r+1})\cdots(1-u^{r+4})(1-u^{k-r+1})\cdots(1-u^{k-r+4})}{(1-u)(1-u^2)^2\cdots(1-u^4)^2}.
\end{aligned}
$$

Let $R(u)=\sum_{r=0}^{k}\frac{(1-u^{r+1})\cdots(1-u^{r+4})(1-u^{k-r+1})\cdots(1-u^{k-r+4})}{(1-u)(1-u^2)^2\cdots(1-u^4)^2}$, if $P(t)=\sum_{k=0}^{+\infty}M(4,k)t^k$, then $M(4,k)=Res_{q=0}q^{-4k-1}R(u)$. Analogue to the proof of Caykey-Sylvester formula(cf. [13]), we have

Proposition 5.12. *The subspace of degree k of the invariant ring $P_0^{sl(2,\mathbb{C})}$ has dimension*

$$
M(4,k)=[R(u)]_{2k}. \tag{5.10}
$$

Using above proposition, we calculate to get

$$
\begin{aligned}
&M(4,0)=1, M(4,1)=0, M(4,2)=3, M(4,3)=4,\\
&M(4,4)=7, M(4,5)=12, M(4,6)=23, \cdots\cdots
\end{aligned}
$$

Remark 5.13. $M(4,2)=3$ shows that the subspace of degree 2 in $P_0^{sl(2,\mathbb{C})}$ is 3-dimensional, $M(4,3)=4$ shows that the subspace of degree 3 is 4-dimensional. According to Remark 5.9, we know that $P_0^{sl(2,\mathbb{C})}$ has three generators of degree 2 and four generators of degree 3, So if we take a basis of the subspace of degree 2 and the subspace of degree 3, respectively, then given two base can be viewed as generators of $P_0^{sl(2,\mathbb{C})}$. Since invariants of degree 2 can generate invariants with degree 4 by multiplication. For given an appropriate basis of the subspace of degree 2, they can generate six linearly independent invariants with degree 4. However, $M(4,4)=7$ shows that the subspace of degree 4 in $P_0^{sl(2,\mathbb{C})}$ is 7-dimensional, we can find an invariant of degree 4 such that it can form a basis of the subspace of degree 4 together with six invariants obtained by given basis of the subspace of degree 2. Since eight generators of $P_0^{sl(2,\mathbb{C})}$ satisfy a polynomial relation of degree 12, so the 7-th invariant of degree 4 is a generator.

Next, we shall try to find eight generators of $P_0^{sl(2,\mathbb{C})}$. Denote by $\mathcal{A}=\widehat{\psi}(\mathcal{O}(sl(2,\mathbb{C}),-\frac{5}{8}K)\subset S(V_4)^{\Theta_+}$. Since there are $gr(\mathcal{A})\hookrightarrow gr(S(V_4)^{\Theta_+})$ and $gr(S(V_4)^{\Theta_+})\hookrightarrow gr(S(V_4))^{\Theta_+}$, hence we have $gr(\mathcal{A})\hookrightarrow gr(S(V_4))^{\Theta_+}=P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$. By Lemma 3.8, we know

$$
v^e=\phi_2(v^e(z))=\gamma_0^{\xi_0}\gamma_0^{\xi_4}-\frac{1}{4}\gamma_0^{\xi_1}\gamma_0^{\xi_3}+\frac{1}{12}\gamma_0^{\xi_2}\gamma_0^{\xi_2},
$$

$$v^f = \phi_2(v^f(z)) = 4\beta_0^{\varphi_1}\beta_0^{\varphi_3} - \beta_0^{\varphi_0}\beta_0^{\varphi_4} - 3\beta_0^{\varphi_2}\beta_0^{\varphi_2},$$

$$v^h = \phi_2(v^h(z)) = \sum_{i=0}^{4} \beta_0^{\varphi_i}\gamma_0^{\xi_i}$$

lie in the invariant ring $P_0^{sl(2,\mathbb{C})}$. Obviously, there are all homogeneous elements of degree 2 in $P_0^{sl(2,\mathbb{C})}$.

Proposition 5.14. *v^e, v^f, v^h are algebraically independent.*

Proof. If we restrict to the subspace $\gamma_0^{\xi_0} = \gamma_0^{\xi_4} = 0, \gamma_0^{\xi_1} = \gamma_0^{\xi_3}, \beta_0^{\varphi_0} = \beta_0^{\varphi_4}, \beta_0^{\varphi_2} = \gamma_0^{\xi_2}$, then v^e, v^f, v^h become $v_r^e = -\frac{1}{4}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + \frac{1}{12}\gamma_0^{\xi_2}\gamma_0^{\xi_2}, v_r^f = -\beta_0^{\varphi_0}\beta_0^{\varphi_0} - 3\gamma_0^{\xi_2}\gamma_0^{\xi_2}, v_r^h = \gamma_0^{\xi_2}\gamma_0^{\xi_2}$, define a map

$$\begin{aligned} \sigma: \qquad & \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \\ (\gamma_0^{\xi_1}, \beta_0^{\varphi_0}, \gamma_0^{\xi_2}) & \longmapsto (v_r^e, v_r^f, v_r^h). \end{aligned}$$

Since σ has been separated variables, so it is a surjective, and v_r^e, v_r^f, v_r^h cannot satisfy any polynomial identity. Hence v^e, v^f, v^h are algebraically independent.

Proposition 5.15. *For the invariant ring $P_0^{sl(2,\mathbb{C})}$, there are four generators of degree 3 as follows*

$$I_1 = 8\gamma_0^{\xi_0}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - 3\gamma_0^{\xi_0}\gamma_0^{\xi_3}\gamma_0^{\xi_3} - 3\gamma_0^{\xi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + \gamma_0^{\xi_1}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - \tfrac{2}{9}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2},$$

$$I_2 = \beta_0^{\varphi_0}\beta_0^{\varphi_2}\beta_0^{\varphi_4} - \beta_0^{\varphi_0}\beta_0^{\varphi_3}\beta_0^{\varphi_3} - \beta_0^{\varphi_1}\beta_0^{\varphi_1}\beta_0^{\varphi_4} + 2\beta_0^{\varphi_1}\beta_0^{\varphi_2}\beta_0^{\varphi_3} - \beta_0^{\varphi_2}\beta_0^{\varphi_2}\beta_0^{\varphi_2},$$

$$\begin{aligned} I_3 = {} & \beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0} - \beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0} - \frac{1}{2}\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1} + \frac{1}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2} \\ & + \frac{1}{2}\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_3} - \frac{1}{2}\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2} - \frac{1}{2}\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_3} + \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_4} \\ & - \beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_4} + \frac{1}{2}\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1} + \frac{1}{6}\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}, \end{aligned}$$

$$\begin{aligned} I_4 = {} & \beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + 3\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + 6\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \frac{3}{8}\beta_0^{\varphi_0}\gamma_0^{\xi_1}\gamma_0^{\xi_1} \\ & - \frac{1}{2}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + \frac{3}{4}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + 3\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} - \frac{1}{2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} \\ & - \frac{1}{2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + \beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - \frac{3}{8}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3}. \end{aligned}$$

Proof. The action of $sl(2,\mathbb{C}) \otimes \mathbb{C}[t]$ on P as derivations is given by the relation (4.4), which induces the action of $sl(2,\mathbb{C})$ on P_0, the action is given by the relation (4.4) in the case of $n = 0$, denote by $\widehat{u}(0)$ for $u \in sl(2,\mathbb{C})$. Actually, the action rules are the relations (2.2) and (2.3).

For I_1, I_2, using the relations (4.4), (2.2) and (2.3), we can check easily they belong to the invariant ring $P_0^{sl(2,\mathbb{C})}$.

Here, we check the I_3 is an invariant in P_0, we need only to check the actions of generators $\{e, f, h\}$ by the linearity of Lie algebra $sl(2, \mathbb{C})$. Firstly, according to the relations (4.4), (2.2) and (2.3), there are

$$\widehat{h}(0)(\beta_0^{\varphi_i}) = (4-2i)\beta_0^{\varphi_i}, \widehat{h}(0)(\gamma_0^{\xi_i}) = -(4-2i)\gamma_0^{\xi_i},$$

so an invariant consisting of product of two β's and one $\gamma's$ should be the linear combination of such some monomials

$$\{\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}, \beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}, \beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}, \beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}, \beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_3}, \beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}, \\ \beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_3}, \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_4}, \beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_4}, \beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}, \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\}.$$

Assume that

$$\begin{aligned} I_3 = {} & k_0\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0} + k_1\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0} + k_2\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1} + k_3\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2} + k_4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_3} \\ & + k_5\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2} + k_6\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_3} + k_7\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_4} + k_8\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_4} + k_9\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1} \\ & + k_{10}\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}, \end{aligned}$$

using the relations

$$\widehat{e}(0)(\beta_0^{\varphi_i}) = i\beta_0^{\varphi_{i-1}}, \widehat{e}(0)(\gamma_0^{\xi_i}) = -(i+1)\gamma_0^{\xi_{i+1}},$$

then there is

$$\begin{aligned} \widehat{e}(0)(I_3) = {} & k_0(2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0} - \beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}) + k_1(2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0} - \beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}) \\ & + k_2(\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1} + 2\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_2}) + k_3(\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_2} \\ & + 3\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_3}) + k_4(\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_3} + 4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_3} \\ & - 4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_4}) + k_5(4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_3}) + k_6(2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_3} \\ & + 3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_4}) + k_7(2\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_4} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_4}) \\ & + 6k_8\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_4} + k_9(3\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1} - 2\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_2}) + k_{10}(4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_2} \\ & - 3\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_3}), \end{aligned}$$

if $\widehat{e}(0)(I_3) = 0$, i.e.

$$\begin{cases} 2k_1 + 2k_0 = 0, \\ -k_1 + 2k_2 = 0, \\ k_2 - k_0 + 3k_9 = 0, \\ -2k_2 + 3k_3 + 4k_5 = 0, \\ k_3 - 2k_9 + 4k_{10} = 0, \\ -3k_3 + 4k_4 + 2k_6 = 0, \\ k_4 - 3k_{10} = 0, \\ -4k_4 + 2k_7 = 0, \\ -3k_5 + 3k_6 = 0, \\ -4k_6 + 4k_7 + 6k_8 = 0. \end{cases}$$

Taking $k_0 = 1$, and solving the system of equations, we can get

$$k_0 = 1, k_1 = -1, k_2 = -\frac{1}{2}, k_3 = \frac{1}{3}, k_4 = \frac{1}{2}, k_5 = -\frac{1}{2},$$
$$k_6 = -\frac{1}{2}, k_7 = 1, k_8 = -1, k_9 = \frac{1}{2}, k_{10} = \frac{1}{6}.$$

Using the relations

$$\widehat{f}(0)(\beta_0^{\varphi_i}) = (4-i)\beta_0^{\varphi_{i+1}}; \widehat{f}(0)(\gamma_0^{\xi_i}) = -(5-i)\gamma_0^{\xi_{i-1}},$$

we can check that

$$\begin{aligned}
\widehat{f}(0)(I_3) = {} & (4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0} + 2\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}) - 6\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0} - \frac{1}{2}(3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1} \\
& + 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1} - 4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}) + \frac{1}{3}(3\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2} + \beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2} \\
& - 3\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}) + \frac{1}{2}(3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}) - \frac{1}{2}(4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2} \\
& - 3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}) - \frac{1}{2}(2\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3} + \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}) \\
& + (2\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_4} - \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}) - (2\beta_0^{\varphi_4}\beta_0^{\varphi_3}\gamma_0^{\xi_4} - \beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}) \\
& + \frac{1}{2}(4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1} + \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}) + \frac{1}{6}(4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2} \\
& - 3\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}) \\
= {} & 0,
\end{aligned}$$

finally, we check to know that $I_3 \in P_0^{sl(2,\mathbb{C})}$.

Similarly, we can check that an invariant consisting of product of one β's and two $\gamma's$ should be multiples of I_4 given in this proposition.

Using the relations (4.4), (2.2) and (2.3), we have

$$\widehat{h}(0)(\beta_0^{\varphi_i}) = (4-2i)\beta_0^{\varphi_i}; \widehat{h}(0)(\gamma_0^{\xi_i}) = -(4-2i)\gamma_0^{\xi_i},$$

so an invariant consisting of product of one β's and two $\gamma's$ should be the linear combination of such some monomials

$$\left\{ \begin{array}{l} \beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_2}, \beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_3}, \beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4}, \beta_0^{\varphi_0}\gamma_0^{\xi_1}\gamma_0^{\xi_1}, \beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_2}, \beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3}, \\ \beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4}, \beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2}, \beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3}, \beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4}, \beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3} \end{array} \right\}$$

Assume that

$$\begin{aligned}
I_4 = {} & k_0\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + k_1\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + k_2\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + k_3\beta_0^{\varphi_0}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + k_4\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_2} \\
& + k_5\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + k_6\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + k_7\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + k_8\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + k_9\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} \\
& + k_{10}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3},
\end{aligned}$$

according to the relations

$$\widehat{e}(0)(\beta_0^{\varphi_i}) = i\beta_0^{\varphi_{i-1}}, \widehat{e}(0)(\gamma_0^{\xi_i}) = -(i+1)\gamma_0^{\xi_{i+1}},$$

then there is

$$\begin{aligned}\widehat{e}(0)(I_4) = &k_0(-\beta_0^{\varphi_0}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_3}) + k_1(-\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\ &- 4\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + \beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_3}) + k_2(-\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_4}\\ &+ 2\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_4}) - 4k_3\beta_0^{\varphi_0}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + k_4(-2\beta_0^{\varphi_1}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\\ &- 3\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + \beta_0^{\varphi_0}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) + k_5(-2\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_3}\\ &- 4\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + 2\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_3}) + k_6(-2\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4}\\ &+ 3\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_4}) + k_7(-6\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + 2\beta_0^{\varphi_1}\gamma_0^{\xi_2}\gamma_0^{\xi_2})\\ &+ k_8(-3\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4} + 3\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_3})\\ &+ k_9(-3\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4} + 4\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4}) + k_{10}(-8\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4}\\ &+ 4\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3}),\end{aligned}$$

if $\widehat{e}(0)(I_4) = 0$, i.e.

$$\begin{cases} -k_0 - 4k_3 + k_4 = 0, \\ -3k_0 + k_1 = 0, \\ -k_1 - 3k_4 + 2k_5 = 0, \\ -4k_1 + 2k_2 = 0, \\ -k_2 - 4k_5 + 3k_6 = 0, \\ -2k_4 + 2k_7 = 0, \\ -2k_5 - 6k_7 + 3k_8 = 0, \\ -2k_6 - 4k_8 + 4k_9 = 0, \\ -3k_8 + 4k_{10} = 0, \\ -3k_9 - 8k_{10} = 0. \end{cases}$$

Taking $k_0 = 1$, and solving the system of equations, we can get

$$k_0 = 1, k_1 = 3, k_2 = 6, k_3 = -\frac{3}{8}, k_4 = -\frac{1}{2}, k_5 = \frac{3}{4}, k_6 = 3,$$
$$k_7 = -\frac{1}{2}, k_8 = -\frac{1}{2}, k_9 = 1, k_{10} = -\frac{3}{8}.$$

Using the relations

$$\widehat{f}(0)(\beta_0^{\varphi_i}) = (4-i)\beta_0^{\varphi_{i+1}}, \widehat{f}(0)(\gamma_0^{\xi_i}) = -(5-i)\gamma_0^{\xi_{i-1}},$$

we can check that

$$\begin{aligned}\widehat{f}(0)(I_4) &= (-3\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_1} + 4\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2}) + 3(-2\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + 3\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3})\\
&+ 6(-\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + 2\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4}) - \frac{3}{8}(-8\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_1} + 4\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1})\\
&- \frac{1}{2}(-4\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + 3\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) + \frac{3}{4}(-4\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3}\\
&- 2\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + 2\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3}) + 3(-4\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\
&+ \beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4}) - \frac{1}{2}(-6\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + 2\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2}) - \frac{1}{2}(-3\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\
&- 2\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + \beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) + (-3\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} - \beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3})\\
&+ \frac{3}{2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3}\\
&= 0,\end{aligned}$$

finally, we check to know that $I_4 \in P_0^{sl(2,\mathbb{C})}$.

Since the subspace of invariants with degree 4 in $P_0^{sl(2,\mathbb{C})}$ is 7− dimensional. However, $\{v^e v^e, v^f v^f, v^h v^h, v^e v^f, v^f v^h, v^e v^h\}$ is a set of six linear independent invariants with degree 4, the following is 7-th invariant with degree 4 as a generator of $P_0^{sl(2,\mathbb{C})}$.

Proposition 5.16.

$$\begin{aligned}\mathcal{H} = &-\frac{8}{3}\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + \beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3}\\
&+ \frac{2}{3}\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + \frac{1}{9}\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\\
&+ \frac{8}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} - \beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + 4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3}\\
&- \frac{2}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + \frac{8}{9}\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\\
&- 4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + \frac{2}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4}\\
&+ 2\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - \beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4}\\
&- \frac{2}{3}\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - \frac{8}{3}\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} + \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3}\\
&+ \frac{8}{3}\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - \beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3}\end{aligned}$$

is a generator with degree 4 of $P_0^{sl(2,\mathbb{C})}$.

Proof. Using the relations

$$\widehat{h}(0)(\beta_0^{\varphi_i}) = (4-2i)\beta_0^{\varphi_i}, \widehat{h}(0)(\gamma_0^{\xi_i}) = -(4-2i)\gamma_0^{\xi_i},$$

we consider an invariant with degree 4 consisting of product of two β's and two $\gamma's$, it should be the linear combination of such some monomials

$$\left\{\begin{array}{l}\beta_0^{\varphi_0}\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_0},\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_1},\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2},\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1},\\ \beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3},\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2},\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_4},\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_3},\\ \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2},\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2},\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1},\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3},\\ \beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2},\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4},\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3},\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2},\\ \beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4},\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3},\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4},\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3},\\ \beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2},\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4},\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3},\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4},\\ \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3},\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4},\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3},\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4},\\ \beta_0^{\varphi_4}\beta_0^{\varphi_4}\gamma_0^{\xi_4}\gamma_0^{\xi_4}\end{array}\right\}$$

We suppose that

$$\begin{aligned}H = {} & k_0\beta_0^{\varphi_0}\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_0} + k_1\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_1} + k_2\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + k_3\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1}\\ & + k_4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + k_5\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + k_6\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + k_7\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\ & + k_8\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + k_9\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + k_{10}\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + k_{11}\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3}\\ & + k_{12}\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + k_{13}\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + k_{14}\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + k_{15}\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\\ & + k_{16}\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + k_{17}\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + k_{18}\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + k_{19}\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\ & + k_{20}\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + k_{21}\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + k_{22}\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + k_{23}\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4}\\ & + k_{24}\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3} + k_{25}\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4} + k_{26}\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3} + k_{27}\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4}\\ & + k_{28}\beta_0^{\varphi_4}\beta_0^{\varphi_4}\gamma_0^{\xi_4}\gamma_0^{\xi_4},\end{aligned}$$

using the relations

$$\widehat{e}(0)(\beta_0^{\varphi_i}) = i\beta_0^{\varphi_{i-1}}, \widehat{e}(0)(\gamma_0^{\xi_i}) = -(i+1)\gamma_0^{\xi_{i+1}},$$

if $\widehat{e}(0)(H) = 0$, i.e.

$$\begin{aligned}\widehat{e}(0)(H) = {} & -2k_0\beta_0^{\varphi_0}\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_1} + k_1(\beta_0^{\varphi_0}\beta_0^{\varphi_0}\gamma_0^{\xi_0}\gamma_0^{\xi_1} - \beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1}\\ & - 2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2}) + k_2(2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} - \beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2}\\ & - 3\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3}) + k_3(2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2})\\ & + k_4(3\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3} - \beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4})\\ & + k_5(3\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 2\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3})\end{aligned}$$

$$
\begin{aligned}
&+ k_6(4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4}) + k_7(4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\
&- 2\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4}) + k_8(4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\\
&- 6\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) + k_9(2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} - \beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_2}\\
&- 3\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_3}) + k_{10}(2\beta_0^{\varphi_0}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1} - 4\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_2})\\
&+ k_{11}(\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + 2\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_3} - \beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\
&- 4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4}) + k_{12}(\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + 2\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_2}\\
&- 2\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3}) + k_{13}(\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4}\\
&+ 3\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4}) + k_{14}(\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3}\\
&+ 3\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4})\\
&+ k_{15}(\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + 3\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 6\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3})\\
&+ k_{16}(\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + 4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4})\\
&+ k_{17}(\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + 4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - 3\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3}\\
&- 4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4}) + k_{18}(4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_4})\\
&+ k_{19}(4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_4})\\
&+ k_{20}(4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 6\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) + k_{21}(2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4}\\
&+ 3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_4} - 2\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4}) + k_{22}(2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3}\\
&+ 3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - 3\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4})\\
&+ k_{23}(2\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - 3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4})\\
&+ k_{24}(2\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3} - 8\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4})\\
&+ k_{25}(6\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - 3\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_4}) + k_{26}(6\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3}\\
&- 8\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_4}) + k_{27}(3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_4} + 4\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_4}\\
&- 4\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_4}\gamma_0^{\xi_4}) + 8k_{28}\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_4}\gamma_0^{\xi_4}\\
&= 0,
\end{aligned}
$$

then we get the system of equations

$$
\begin{cases}
-2k_0+k_1=0\\
-k_1+2k_3+2k_{10}=0,\\
-2k_1+2k_2+2k_9=0,\\
-k_2-4k_3+3k_5+k_{12}=0,\\
-3k_2+3k_4+k_{11}=0,\\
-k_4-3k_5+4k_7+k_{14}=0,\\
-4k_4+4k_6+k_{13}=0,\\
-2k_5+4k_8+k_{15}=0,\\
-k_6-4k_7+k_{16}=0,\\
-2k_7-6k_8+k_{17}=0,\\
-k_9-4k_{10}+2k_{12}=0,\\
\ \ 2k_{11}-3k_9=0,\\
-k_{11}-3k_{12}+3k_{14}+4k_{19}=0,\\
-4k_{11}+3k_{13}+4k_{18}=0,\\
-2k_{12}+4k_{20}+3k_{15}=0,\\
-4k_{14}+4k_{16}+2k_{21}=0,\\
-2k_{14}-6k_{15}+4k_{17}+2k_{22}=0,\\
-2k_{16}-4k_{17}+2k_{23}=0,\\
-3k_{17}+2k_{24}=0,\\
-k_{18}-4k_{19}+3k_{21}=0,\\
-2k_{19}-6k_{20}+3k_{22}=0,\\
-2k_{21}-4k_{22}+4k_{23}+6k_{25}=0,\\
-3k_{22}+4k_{24}+6k_{26}=0,\\
-3k_{23}-8k_{24}+3k_{27}=0,\\
-3k_{25}-8k_{26}+4k_{27}=0,\\
-4k_{27}+8k_{28}=0.
\end{cases}
$$

Above system of equations has 29 indeterminates and 26 equations, so its solutions has three free indeterminates, we choose k_0, k_3, k_{13} are the free indeterminates, we can know that the space of solutions of the system of equations is 3-dimensional. Firstly, we choose $k_0=0, k_3=0, k_{13}=1$, then the solution of above system of equations is

$$
\begin{aligned}
&k_0=k_1=k_2=k_3=k_4=k_5=0, k_6=-\frac{1}{4}, k_7=\frac{1}{16}, k_8=-\frac{1}{48},\\
&k_9=k_{10}=k_{11}=k_{12}=0, k_{13}=1, k_{14}=-\frac{1}{4}, k_{15}=\frac{1}{12}, k_{16}=k_{17}=0,
\end{aligned}
$$

$$k_{18} = -\frac{3}{4}, k_{19} = \frac{3}{16}, k_{20} = -\frac{1}{16}, k_{21} = k_{22} = k_{23} = k_{24} = k_{25} = k_{26} = k_{27} = k_{28} = 0,$$

the corresponding element H is $\frac{1}{4}v^e v^f$, which is an invariant in $P_0^{sl(2,\mathbb{C})}$.

If we choose $k_0 = 1, k_3 = 0, k_{13} = 0$, the solution of above system of equations is

$$\begin{aligned}
&k_0 = 1, k_1 = 2, k_2 = 2, k_3 = 0, k_4 = 2, k_5 = 0, k_6 = 2, k_7 = k_8 = k_9 = 0,\\
&k_{10} = 1, k_{11} = 0, k_{12} = 2, k_{13} = 0, k_{14} = 2, k_{15} = 0, k_{16} = 2, k_{17} = k_{18}\\
&= k_{19} = 0, k_{20} = 1, k_{21} = 0, k_{22} = 2, k_{23} = 2, k_{24} = k_{25} = 0, k_{26} = 1,\\
&k_{27} = 2, k_{28} = 1,
\end{aligned}$$

the corresponding element H is $v^h v^h$, which is an invariant in $P_0^{sl(2,\mathbb{C})}$.

Finally, we take $k_0 = 0, k_3 = 1, k_{13} = 0$, then the solution of above system of equations is

$$\begin{aligned}
&k_0 = k_1 = 0, k_2 = -\frac{8}{3}, k_3 = 1, k_4 = -4, k_5 = \frac{2}{3}, k_6 = -4, k_7 = 0,\\
&k_8 = \frac{1}{9}, k_9 = \frac{8}{3}, k_{10} = -1, k_{11} = 4, k_{12} = -\frac{2}{3}, k_{13} = 0, k_{14} = -2,\\
&k_{15} = \frac{8}{9}, k_{16} = -4, k_{17} = \frac{2}{3}, k_{18} = 4, k_{19} = 2, k_{20} = -1, k_{21} = 4,\\
&k_{22} = -\frac{2}{3}, k_{23} = -\frac{8}{3}, k_{24} = 1, k_{25} = \frac{8}{3}, k_{26} = -1, k_{27} = 0, k_{28} = 0.
\end{aligned}$$

Using the relations

$$\widehat{f}(0)(\beta_0^{\varphi_i}) = (4-i)\beta_0^{\varphi_{i+1}}, \widehat{f}(0)(\gamma_0^{\xi_i}) = -(5-i)\gamma_0^{\xi_{i-1}},$$

we can check corresponding element $\mathcal{H}$ with degree 4 satisfies

$$\begin{aligned}
\widehat{f}(0)(\mathcal{H}) = &-\frac{8}{3}(4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + 2\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_2} - 3\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_1})\\
&+ (4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + 2\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_1} - 8\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_1})\\
&- 4(4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_2})\\
&+ \frac{2}{3}(4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_2}\\
&- 3\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_1}) - 4(4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_3})\\
&+ \frac{1}{9}(4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 6\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) + \frac{8}{3}(6\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2}\\
&- 3\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_1}) - (6\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1} - 8\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_1})\\
&+ 4(3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3} + 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2})\\
&- \frac{2}{3}(3\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} + 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2}
\end{aligned}$$

$$
\begin{aligned}
&- 3\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1}) - 2(3\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + \beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_3} \\
&- 4\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) + \frac{8}{9}(3\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2} \\
&+ \beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 6\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) - 4(3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} \\
&- 4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_4} - \beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_3}) + \frac{2}{3}(3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} \\
&- 3\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2}) + 4(4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4} \\
&- \beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3}) + 2(4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3} \\
&- 2\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) - (4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2} - 6\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2}) \\
&+ 4(2\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} - 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_4} \\
&- \beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3}) - \frac{2}{3}(2\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} \\
&- 3\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - 2\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2}) - \frac{8}{3}(2\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} \\
&- 3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} - 3\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) + (2\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3} \\
&- 4\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) + \frac{8}{3}(2\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - 3\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} \\
&- \beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) - (2\beta_0^{\varphi_3}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3} - 4\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3}) \\
&= 0,
\end{aligned}
$$

so

$$
\begin{aligned}
\mathcal{H} = &-\frac{8}{3}\beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_2} + \beta_0^{\varphi_0}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_1} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_0}\gamma_0^{\xi_3} \\
&+ \frac{2}{3}\beta_0^{\varphi_0}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 4\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_0}\gamma_0^{\xi_4} + \frac{1}{9}\beta_0^{\varphi_0}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_2} \\
&+ \frac{8}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_0}\gamma_0^{\xi_2} - \beta_0^{\varphi_1}\beta_0^{\varphi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_1} + 4\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_3} \\
&- \frac{2}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_2} - 2\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_3} + \frac{8}{9}\beta_0^{\varphi_1}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_2} \\
&- 4\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_1}\gamma_0^{\xi_4} + \frac{2}{3}\beta_0^{\varphi_1}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_3} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_0}\gamma_0^{\xi_4} \\
&+ 2\beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_1}\gamma_0^{\xi_3} - \beta_0^{\varphi_2}\beta_0^{\varphi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} + 4\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_1}\gamma_0^{\xi_4} \\
&- \frac{2}{3}\beta_0^{\varphi_2}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_3} - \frac{8}{3}\beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_2}\gamma_0^{\xi_4} + \beta_0^{\varphi_2}\beta_0^{\varphi_4}\gamma_0^{\xi_3}\gamma_0^{\xi_3} \\
&+ \frac{8}{3}\beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_2}\gamma_0^{\xi_4} - \beta_0^{\varphi_3}\beta_0^{\varphi_3}\gamma_0^{\xi_3}\gamma_0^{\xi_3}
\end{aligned}
$$

is an invariant of $P_0^{sl(2,\mathbb{C})}$. According to Remark 5.13, we know the invariant $\mathcal{H}$ is a generator of degree 4 of $P_0^{sl(2,\mathbb{C})}$.

Theorem 5.17. *The invariant ring $P_0^{sl(2,\mathbb{C})}$ is generated by the finite set $\{v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H}\}$. And these generators subject to a polynomial* *relation of degree 12, denote by*

$$F(v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H}) = 0,$$

i.e.

$$P_0^{sl(2,\mathbb{C})} = \mathbb{C}[v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H}]/\langle F(v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H})\rangle,$$

where $\langle F(v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H})\rangle$ means the ideal generated by polynomial $F(v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H})$.

According to Lemma 4.4, we know

Corollary 5.18. *The invariant ring $P^{\Theta_+} = P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by the finite set $\{v^e, v^f, v^h, I_1, I_2, I_3, I_4, \mathcal{H}\}$ as $\partial-$ ring.*

6 The Generators of Vertex Algebra Commutant $S(V_4)^{\Theta_+}$

In this section, we shall give a description clearly of $S(V_4)^{\Theta_+}$ by giving its generators.

We have known that $\Gamma : gr(S(V_4)^{\Theta_+}) \hookrightarrow P^{\Theta_+}$, for each generator of P^{Θ_+}, if we can find an element in $gr(S(V_4)^{\Theta_+})$ such that it is an inverse image of this generator under the map Γ, then we can determine the map Γ is surjective, therefore, we can describe $S(V_4)^{\Theta_+}$ by the reconstruction property of $\partial-$ rings.

Proposition 6.1. *Three generators $\{v^e, v^f, v^h\}$ of P^{Θ_+} lie in the subalgebra $gr(S(V_4)^{\Theta_+})$.*

Proof. By Corollary 3.9, we know that $v^e(z), v^f(z), v^h(z)$ belong to the commutant $S(V_4)^{\Theta_+}$. Since

$$\phi_2(v^e(z)) = v^e, \phi_2(v^f(z)) = v^f, \phi_2(v^h(z)) = v^h,$$

so $\{v^e, v^f, v^h\}$ belong to the $\partial-$ ring $gr(S(V_4)^{\Theta_+})$.

Next we need to determine whether $I_1(z), I_2(z), I_3(z), I_4(z)$ lie in $S(V_4)^{\Theta_+}$. According to the definition of (3.8) of vertex algebra commutant, for a vertex operator $v(z) \in S(V_4)$, if for all $u(z) \in \Theta$, the negative circle product $u(z) \circ_n v(z) = 0$, then $v(z) \in S(V_4)^{\Theta_+}$. Using the relation (3.1), we know that this condition is equivalent to the OPE relations $u(z)v(w) \sim 0$. Furthermore, since Θ is generated by $\widehat{e}(z), \widehat{f}(z), \widehat{h}(z)$, therefore, the conditions

$$\begin{cases} \widehat{e}(z)v(w) \sim 0, \\ \widehat{f}(z)v(w) \sim 0, \\ \widehat{h}(z)v(w) \sim 0 \end{cases}$$

can guarantee the OPE relations $u(z)v(w) \sim 0$ are satisfied. Therefore, we only need to calculate OPE relations with generated vertex operators $\widehat{e}(z), \widehat{f}(z), \widehat{h}(z)$ for our goal mentioned in the beginning of this paragraph.

For the convenience of calculating OPE relations, we need the help of the following two lemmas in [6]. Let $a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$ be a field, we denote by $a(z)_- = \sum_{n\geq 0} a(n)z^{-n-1}, a(z)_+ = \sum_{n<0} a(n)z^{-n-1}$, and there are $\partial a(z)_\pm = \partial(a(z)_\pm)$.

Lemma 6.2. *([6]) For any field $a(z,w)$ and any positive integer N, there exist field $c^j(w)(0 \leq j \leq N-1)$ and a field $d^N(z,w)$ such that*

$$a(z,w) = \sum_{j=0}^{N-1} c^j(w)(z-w)^j + (z-w)^N d^N(z,w). \tag{6.1}$$

The coefficients $c^j(w)$ are uniquely determined by this expansion and are given by the usual formula:

$$c^j(w) = \partial_z^{(j)} a(z,w)|_{z=w}.$$

The following well-known theorem is extremely useful for the OPE of two normally ordered products of "free" fields.

Lemma 6.3. *(Wick theorem)([6]) Let $a^1(z), \cdots, a^M(z)$ and $b^1(z), \cdots, b^N(z)$ be two collections of fields such that the following properties hold:*

(1) $[[a^i(z)_-, b^j(w)], c^k(z)_\pm] = 0$ for all i, j, k, and $c = a$, or b.

(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ for all i and j.

Let $[a^i, b^j] = [a^i(z)_-, b^j(w)]$ denote the "contraction" of $a^i(z)$ and $b^j(w)$. Then one has:

$$: a^1(z)\cdots a^M(z) :: b^1(w)\cdots b^N(w) := \sum_{s=0}^{min(M,N)} \sum_{i_1<\cdots<i_s; j_1\neq\cdots\neq j_s}$$
$$(\pm[a^{i_1}b^{j_1}]\cdots[a^{i_s}b^{j_s}] : a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w)) :_{(i_1,\cdots,i_s;j_1,\cdots,j_s)} \tag{6.2}$$

where the subscript $(i_1,\cdots,i_s;j_1,\cdots,j_s)$ means that the fields $a^{i_1}(z),\cdots,a^{i_s}(z)$, $b^{j_1}(z),\cdots,b^{j_s}(z)$ are removed, and the sign $\pm$ is obtained by the usual superrule: each permutation of the adjacent odd fields changes the sign.

It is clear that if $[a^i(z)_\pm, a^j(z)_\pm] = 0$ for all $i, j \in \{1, 2, \cdots, N\}$, then $: a^1(z)a^2(z)\cdots a^N(z) := \pm : a^{i_1}(z)a^{i_2}(z)\cdots a^{i_N}(z) :$, where $\pm$ is the sign of the permutation of $i_1, \cdots, i_N$ from which the indices of even fields are removed. Here, all fields considered are even fields, so all signs are 1. It follows that in this case the wick product is commutative. This property will help us calculate OPE relations in latter. Next, we give some useful commutation relations to calculate OPE relations by Lemma 6.2, 6.3.

$$[\beta^{\varphi_i}(z)_\pm, \gamma^{\xi_j}(w)_\pm] = 0, \tag{6.3}$$

$$[\beta^{\varphi_i}(z)_-, \gamma^{\xi_j}(w)] = \frac{\langle \varphi_i, \xi_j \rangle}{z-w} = \frac{\delta_{i,j}}{z-w}, \tag{6.4}$$

$$[\gamma^{\xi_i}(z)_-, \beta^{\varphi_j}(w)] = -\frac{\langle \varphi_i, \xi_j \rangle}{z-w} = -\frac{\delta_{i,j}}{z-w}, \tag{6.5}$$

$$[\beta^{\varphi_i}(z)_-, \beta^{\varphi_j}(w)] = [\gamma^{\xi_i}(z)_-, \gamma^{\xi_j}(w)] = 0, \tag{6.6}$$

$$[\partial\beta^{\varphi_i}(z)_-, \gamma^{\xi_j}(w)] = [\gamma^{\xi_i}(z)_-, \partial\beta^{\varphi_j}(w)] = -\frac{\delta_{i,j}}{(z-w)^2}, \tag{6.7}$$

$$[\partial\gamma^{\xi_i}(z)_-, \beta^{\varphi_j}(w)] = [\beta^{\varphi_i}(z)_-, \partial\gamma^{\xi_j}(w)] = \frac{\delta_{i,j}}{(z-w)^2}, \tag{6.8}$$

$$[\partial\beta^{\varphi_i}(z)_-, \partial\gamma^{\xi_j}(w)] = -\frac{2\delta_{i,j}}{(z-w)^2}, \tag{6.9}$$

$$[\partial\gamma^{\xi_i}(z)_-, \partial\beta^{\varphi_j}(w)] = \frac{2\delta_{i,j}}{(z-w)^2}, \tag{6.10}$$

where $\delta_{i,j}$ means that if $i = j$, the value is 1, otherwise, it is 0.

Next we calculate OPE relations to give the following conclusion.

Proposition 6.4. *all fields $I_1(z), I_2(z), I_3(z), I_4(z)$ lie in $S(V_4)^{\Theta_+}$.*

Proof. According to Proposition 5.4, we know that $I_1(z)$ lies in the conformal weight-zero subspace $S(V_4)_0^{\Theta_+} \subset S(V_4)^{\Theta_+}$.

Next, it is sufficient to check these OPE relations: for $i = 2, 3, 4$,

$$\begin{cases} \widehat{e}(z)I_i(w) \sim 0, \\ \widehat{f}(z)I_i(w) \sim 0, \\ \widehat{h}(z)I_i(w) \sim 0, \end{cases}$$

where there are

$$\widehat{e}(z) = -\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z) :,$$

$$\widehat{f}(z) = -\sum_{i=0}^{3} (4-i) : \beta^{\varphi_{i+1}}(z)\gamma^{\xi_i}(z) :,$$

$$\widehat{h}(z) = -\sum_{i=0}^{4} (4-2i) : \beta^{\varphi_i}(z)\gamma^{\xi_i}(z) : .$$

Firstly, we calculate the OPE relation between $\widehat{e}(z)$ and $I_2(z)$, since

$$\begin{aligned}\widehat{e}(z)I_2(w) = -\sum_{i=1}^{4} i :\beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z) : (: \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w) :\\ - : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w) : - : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w) :\\ + 2 : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) : - : \beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) :)\end{aligned}$$

For each monomial, we have

1)

$$\begin{aligned}\widehat{e}(z) : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w) :&= -\sum_{i=1}^{4} i(: \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w) :\\ &- \frac{\delta_{i,0} : \beta^{\varphi_{i-1}}(z)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w) :}{z-w} - \frac{\delta_{i,2} : \beta^{\varphi_{i-1}}(z)\beta^{\varphi_0}(w)\beta^{\varphi_4}(w) :}{z-w}\\ &- \frac{\delta_{i,4} : \beta^{\varphi_{i-1}}(z)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w) :}{z-w})\\ &= -\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w) :\\ &+ \frac{2 : \beta^{\varphi_1}(z)\beta^{\varphi_0}(w)\beta^{\varphi_4}(w) :}{z-w} + \frac{4 : \beta^{\varphi_3}(z)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w) :}{z-w},\end{aligned}$$

then the OPE relation is

$$\begin{aligned}&\widehat{e}(z) : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w) :\\ &\quad\sim \frac{2 : \beta^{\varphi_1}(w)\beta^{\varphi_0}(w)\beta^{\varphi_4}(w) :}{z-w} + \frac{4 : \beta^{\varphi_3}(w)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w) :}{z-w}.\end{aligned}$$

2)

$$\begin{aligned}\widehat{e}(z) : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w) : = -\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w) :\\ + \frac{6 : \beta^{\varphi_2}(z)\beta^{\varphi_0}(w)\beta^{\varphi_3}(w) :}{z-w},\end{aligned}$$

then there is

$$\widehat{e}(z) : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w) :\sim \frac{6 : \beta^{\varphi_2}(w)\beta^{\varphi_0}(w)\beta^{\varphi_3}(w) :}{z-w}.$$

3)

$$\begin{aligned}\widehat{e}(z) : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w) := -\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\beta^{\varphi_4}(w) :\\ + \frac{2 : \beta^{\varphi_0}(z)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w) :}{z-w} + \frac{4 : \beta^{\varphi_3}(z)\beta^{\varphi_1}(w)\beta^{\varphi_1}(w) :}{z-w},\end{aligned}$$

then there is

$$\widehat{e}(z) : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w) : \\ \sim \frac{2 : \beta^{\varphi_0}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w) :}{z-w} + \frac{4 : \beta^{\varphi_3}(w)\beta^{\varphi_1}(w)\beta^{\varphi_1}(w) :}{z-w}.$$

4)

$$\begin{aligned}&\widehat{e}(z)(2 : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) :)\\ &= -2\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) : + \frac{2 : \beta^{\varphi_0}(z)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) :}{z-w}\\ &+ \frac{4 : \beta^{\varphi_1}(z)\beta^{\varphi_1}(w)\beta^{\varphi_3}(w) :}{z-w} + \frac{6 : \beta^{\varphi_2}(z)\beta^{\varphi_1}(w)\beta^{\varphi_2}(w) :}{z-w},\end{aligned}$$

then there is

$$\begin{aligned}\widehat{e}(z)(2 : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) :) &\sim \frac{2 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) :}{z-w}\\ &+ \frac{4 : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_3}(w) :}{z-w} + \frac{6 : \beta^{\varphi_2}(w)\beta^{\varphi_1}(w)\beta^{\varphi_2}(w) :}{z-w}.\end{aligned}$$

5)

$$\begin{aligned}\widehat{e}(z) : \beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) : = &-\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) :\\ &+ \frac{6 : \beta^{\varphi_1}(z)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) :}{z-w},\end{aligned}$$

then there is

$$\widehat{e}(z) : \beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) :\sim \frac{6 : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) :}{z-w}.$$

Add to OPE relations of 1) $-$ 5), we get

$$\begin{aligned}\widehat{e}(z)I_2(w) \sim &\frac{2 : \beta^{\varphi_1}(w)\beta^{\varphi_0}(w)\beta^{\varphi_4}(w) :}{z-w} + \frac{4 : \beta^{\varphi_3}(w)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w) :}{z-w}\\ &- \frac{6 : \beta^{\varphi_2}(w)\beta^{\varphi_0}(w)\beta^{\varphi_3}(w) :}{z-w} - \frac{2 : \beta^{\varphi_0}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w) :}{z-w}\\ &- \frac{4 : \beta^{\varphi_3}(w)\beta^{\varphi_1}(w)\beta^{\varphi_1}(w) :}{z-w} + \frac{2 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w) :}{z-w}\\ &+ \frac{4 : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_3}(w) :}{z-w} + \frac{6 : \beta^{\varphi_2}(w)\beta^{\varphi_1}(w)\beta^{\varphi_2}(w) :}{z-w}\\ &- \frac{6 : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w) :}{z-w}\\ &= 0.\end{aligned}$$

Next, we calculate the OPE relation between $\widehat{h}(z)$ and $I_2(z)$, since

$$\begin{aligned}\widehat{h}(z)I_2(w) = &-\sum_{i=0}^{4}(4-2i) :\beta^{\varphi_i}(z)\gamma^{\xi_i}(z): (:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w):\\ &-:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w): - :\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w):\\ &+2:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w): - :\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w):),\end{aligned}$$

there is the OPE relation

$$\begin{aligned}\widehat{h}(z)I_2(w) \sim &\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w):}{z-w} - \frac{4:\beta^{\varphi_4}(w)\beta^{\varphi_0}(w)\beta^{\varphi_2}(w):}{z-w}\\ &-\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w):}{z-w} + \frac{4:\beta^{\varphi_3}(w)\beta^{\varphi_0}(w)\beta^{\varphi_3}(w):}{z-w}\\ &-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w):}{z-w} + \frac{4:\beta^{\varphi_4}(w)\beta^{\varphi_1}(w)\beta^{\varphi_1}(w):}{z-w}\\ &+\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w):}{z-w} - \frac{4:\beta^{\varphi_3}(w)\beta^{\varphi_1}(w)\beta^{\varphi_2}(w):}{z-w}\\ &=0.\end{aligned}$$

We calculate the OPE relation between $\widehat{f}(z)$ and $I_2(z)$, since

$$\begin{aligned}\widehat{f}(z)I_2(w) = &-\sum_{i=0}^{3}(4-i) :\beta^{\varphi_{i+1}}(z)\gamma^{\xi_i}(z)(:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w):\\ &-:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w): - :\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w):\\ &+2:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w): - :\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w):),\end{aligned}$$

then there is the following relations

$$\begin{aligned}\widehat{f}(z)I_2(w) \sim &\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\beta^{\varphi_4}(w):}{z-w} + \frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_0}(w)\beta^{\varphi_4}(w):}{z-w}\\ &-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\beta^{\varphi_3}(w):}{z-w} - \frac{2:\beta^{\varphi_4}(w)\beta^{\varphi_0}(w)\beta^{\varphi_3}(w):}{z-w}\\ &-\frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_1}(w)\beta^{\varphi_4}(w):}{z-w} + \frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\beta^{\varphi_3}(w):}{z-w}\\ &+\frac{4:\beta^{\varphi_3}(w)\beta^{\varphi_1}(w)\beta^{\varphi_3}(w):}{z-w} + \frac{2:\beta^{\varphi_4}(w)\beta^{\varphi_1}(w)\beta^{\varphi_2}(w):}{z-w}\\ &-\frac{6:\beta^{\varphi_3}(w)\beta^{\varphi_2}(w)\beta^{\varphi_2}(w):}{z-w}\\ &=0.\end{aligned}$$

Finally, there are

$$\begin{cases}\widehat{e}(z)I_2(w) \sim 0,\\ \widehat{f}(z)I_2(w) \sim 0,\\ \widehat{h}(z)I_2(w) \sim 0,\end{cases}$$

then we know $I_2(z) \in S(V_4)^{\Theta_+}$.

For $I_3(z)$, there are the following OPE relations

$$
\begin{aligned}
\widehat{e}(z)I_3(w) \sim & (\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_0}(w)\gamma^{\xi_0}(w):}{z-w} - \frac{:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{z-w}) \\
& + (\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w):}{z-w} - \frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w):}{z-w}) \\
& - \frac{2\beta^{\varphi_1}(w)}{(z-w)^2} + (\frac{:\beta^{\varphi_1(w)}\beta^{\varphi_2}(w)\gamma^{\xi_2}(w):}{z-w} - \frac{\beta^{\varphi_1}(w)}{(z-w)^2} \\
& - \frac{1}{2}\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{z-w} - \frac{:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w):}{z-w}) \\
& - (\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w):}{z-w} - \frac{1}{3}\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{z-w} \\
& - \frac{:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w):}{z-w}) + \frac{\beta^{\varphi_1}(w)}{(z-w)^2} + \frac{2\beta^{\varphi_1}(w)}{(z-w)^2} \\
& - \frac{1}{2}(\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_4}(w):}{z-w} - \frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w):}{z-w} \\
& - \frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w):}{z-w}) + \frac{1}{2}(\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{z-w} \\
& - \frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w):}{z-w}) + \frac{1}{2}(\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):}{z-w} \\
& - \frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w):}{z-w} - \frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{z-w}) \\
& + (\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_4}(w):}{z-w} + \frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):}{z-w}) \\
& - \frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):}{z-w} - \frac{1}{2}(\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{z-w} \\
& - \frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{z-w}) - \frac{1}{6}(\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w):}{z-w} \\
& - \frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{z-w}) \\
& = 0.
\end{aligned}
$$

$$
\begin{aligned}
\widehat{f}(z)I_3(w) \sim & (\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w):}{z-w} + \frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w):}{z-w}) \\
& - \frac{6:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w):}{z-w} + \frac{1}{2}(\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w):}{z-w} \\
& - \frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{z-w} - \frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w):}{z-w})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{3}(\frac{3 : \beta^{\varphi_1(w)}\beta^{\varphi_3}(w)\gamma^{\xi_1}(w) :}{z-w} - \frac{3 : \beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w) :}{z-w} \\
&- \frac{: \beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w) :}{z-w}) + \frac{\beta^{\varphi_3}(w)}{(z-w)^2} \\
&- \frac{1}{2}(\frac{2 : \beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w) :}{z-w} - \frac{3 : \beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w) :}{z-w}) \\
&+ \frac{1}{2}(\frac{3 : \beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w) :}{z-w} - \frac{4 : \beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w) :}{z-w}) \\
&+ \frac{1}{2}(\frac{2 : \beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w) :}{z-w} - \frac{2 : \beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w) :}{z-w} \\
&- \frac{: \beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w) :}{z-w}) - \frac{\beta^{\varphi_3}(w)}{(z-w)^2} \\
&- (\frac{: \beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w) :}{z-w} - \frac{2 : \beta^{\varphi_3}(w)\beta^{\varphi_4}(w)\gamma^{\xi_4}(w) :}{z-w}) \\
&+ (\frac{: \beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w) :}{z-w} - \frac{2 : \beta^{\varphi_3}(w)\beta^{\varphi_4}(w)\gamma^{\xi_4}(w) :}{z-w}) \\
&- \frac{2\beta^{\varphi_3}(w)}{(z-w)^2} - \frac{1}{2}(\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w) :}{z-w} - \frac{4\beta^{\varphi_3}(w)}{(z-w)^2} \\
&- \frac{4 : \beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w) :}{z-w} - \frac{: \beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w) :}{z-w}) \\
&- \frac{1}{6}(\frac{3 : \beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w) :}{z-w} - \frac{4 : \beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w) :}{z-w}) \\
&= 0.
\end{aligned}
$$

Similarly, for each monomial in $I_3(w)$, we have

$$
\begin{aligned}
\widehat{h}(z) : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w) :\sim & -(\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w) :}{z-w} \\
& - \frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w) :}{z-w} - \frac{4\beta^{\varphi_2}(w)}{(z-w)^2}) \\
& = \frac{4\beta^{\varphi_2}(w)}{(z-w)^2},
\end{aligned}
$$

$$
\begin{aligned}
&\widehat{h}(z)(- : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w) :) \\
&\sim (\frac{4 : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w) :}{z-w} - \frac{4 : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w) :}{z-w}) \\
&= 0,
\end{aligned}
$$

$$
\widehat{h}(z)(-\frac{1}{2} : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w) :) \sim -\frac{\beta^{\varphi_2}(w)}{(z-w)^2},
$$

$$
\widehat{h}(z)(\frac{1}{3} : \beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w) :) \sim 0,
$$

$$\widehat{h}(z)(\frac{1}{2}:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w):)\sim 0,$$
$$\widehat{h}(z)(-\frac{1}{2}:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w):)\sim 0,$$
$$\widehat{h}(z)(-\frac{1}{2}:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w):)\sim \frac{\beta^{\varphi_2}(w)}{(z-w)^2},$$
$$\widehat{h}(z)(:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_4}(w):)\sim -\frac{4\beta^{\varphi_2}(w)}{(z-w)^2},$$
$$\widehat{h}(z)(-:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):)\sim 0,$$
$$\widehat{h}(z)(\frac{1}{2}:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w):)\sim 0,$$
$$\widehat{h}(z)(\frac{1}{6}:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w):)\sim 0.$$

Add to all the monomials of $I_3(w)$, we have

$$\widehat{h}(z)I_3(w)\sim 0.$$

Finally, there are

$$\begin{cases} \widehat{e}(z)I_3(w)\sim 0, \\ \widehat{f}(z)I_3(w)\sim 0, \\ \widehat{h}(z)I_3(w)\sim 0, \end{cases}$$

then we know $I_3(z)\in S(V_4)^{\Theta_+}$.

For $I_4(z)$, using Lemma 6.2 6.3 and similarly calculus, we have

$$\begin{aligned}
\widehat{e}(z)I_4(w)\sim &-(\frac{:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}+\frac{3:\beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}) \\
&-3(\frac{-:\beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}+\frac{:\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w} \\
&+\frac{4:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{\gamma^{\xi_3}(w)}{(z-w)^2}) \\
&-6(\frac{-2:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}+\frac{:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}) \\
&+\frac{3}{2}\frac{:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}+\frac{1}{2}(\frac{-:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w} \\
&+\frac{2:\beta^{\varphi_1}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}+\frac{3:\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}) \\
&-\frac{3}{4}(\frac{-2:\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}+\frac{2:\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w} \\
&+\frac{4:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{2\gamma^{\xi_3}(w)}{(z-w)^2})
\end{aligned}$$

$$
\begin{aligned}
&- 3(\frac{-3 : \beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w) :}{z-w} + \frac{2 : \beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w) :}{z-w}) \\
&+ \frac{1}{2}(\frac{-2 : \beta^{\varphi_1}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w) :}{z-w} + \frac{6 : \beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w) :}{z-w}) \\
&+ \frac{1}{2}(\frac{-3 : \beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w) :}{z-w} + \frac{3 : \beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w) :}{z-w} \\
&+ \frac{4 : \beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w) :}{z-w} - \frac{3\gamma^{\xi_3}(w)}{(z-w)^2}) \\
&- (\frac{-4 : \beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w) :}{z-w} + \frac{3 : \beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_4}(w) :}{z-w}) \\
&+ \frac{3}{8}(\frac{-4 : \beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w) :}{z-w} + \frac{8 : \beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_4}(w) :}{z-w} \\
&- \frac{8\gamma^{\xi_3}(w)}{(z-w)^2}) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\widehat{f}(z)I_4(w) \sim &-(\frac{-4 : \beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :}{z-w} + \frac{3 : \beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_1}(w) :}{z-w}) \\
&- 3(\frac{-3 : \beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w} + \frac{2 : \beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :}{z-w}) \\
&- 6(\frac{-2 : \beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w) :}{z-w} + \frac{: \beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w}) \\
&+ \frac{3}{8}(\frac{-4 : \beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w) :}{z-w} + \frac{8 : \beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_1}(w) :}{z-w} \\
&- \frac{8\gamma^{\xi_1}(w)}{(z-w)^2}) + \frac{1}{2}(\frac{-3 : \beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w) :}{z-w} - \frac{3\gamma^{\xi_1}(w)}{(z-w)^2} \\
&+ \frac{4 : \beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :}{z-w} + \frac{3 : \beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w) :}{z-w}) \\
&- \frac{3}{4}(\frac{-2 : \beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w) :}{z-w} + \frac{4 : \beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w} \\
&+ \frac{2 : \beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w) :}{z-w} - \frac{2\gamma^{\xi_1}(w)}{(z-w)^2}) + \frac{3\gamma^{\xi_1}(w)}{(z-w)^2} \\
&- 3(\frac{- : \beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w) :}{z-w} + \frac{4 : \beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w) :}{z-w} \\
&- \frac{: \beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w) :}{z-w}) + \frac{1}{2}(\frac{-2 : \beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w) :}{z-w} \\
&+ \frac{6 : \beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w) :}{z-w}) + \frac{1}{2}(\frac{- : \beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w) :}{z-w} \\
&+ \frac{3 : \beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w) :}{z-w} + \frac{2 : \beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w) :}{z-w})
\end{aligned}
$$

$$
\begin{aligned}
&-(\frac{-3:\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}+\frac{:\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w})\\
&+\frac{3}{2}\frac{:\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}\\
&=0,
\end{aligned}
$$

for each monomial in $I_4(w)$, there are OPE relations with $\widehat{h}(z)$

$$
\begin{aligned}
&\widehat{h}(z):\beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):\sim-(\frac{4:\beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}\\
&-\frac{4:\beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{4\gamma^{\xi_2}(w)}{(z-w)^2})\\
&=\frac{4\gamma^{\xi_2}(w)}{(z-w)^2},\\
&\widehat{h}(z)(3:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):)\sim-3(\frac{4:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}\\
&-\frac{2:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w})\\
&=0,\\
&\widehat{h}(z)(6:\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):)\sim 0,\\
&\widehat{h}(z)(-\frac{3}{8}:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):)\sim 0,\\
&\widehat{h}(z)(-\frac{1}{2}:\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):)\sim-\frac{\gamma^{\xi_2}(w)}{(z-w)^2},\\
&\widehat{h}(z)(\frac{3}{4}:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):)\sim 0,\\
&\widehat{h}(z)(3:\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):)\sim 0,\\
&\widehat{h}(z)(-\frac{1}{2}:\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):)\sim 0,\\
&\widehat{h}(z)(-\frac{1}{2}:\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):)\sim\frac{\gamma^{\xi_2}(w)}{(z-w)^2},\\
&\widehat{h}(z)(:\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim-\frac{4\gamma^{\xi_2}(w)}{(z-w)^2},\\
&\widehat{h}(z)(-\frac{3}{8}:\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim 0,
\end{aligned}
$$

finally, we get

$$
\begin{cases}
\widehat{e}(z)I_4(w)\sim 0,\\
\widehat{f}(z)I_4(w)\sim 0,\\
\widehat{h}(z)I_4(w)\sim 0,
\end{cases}
$$

then we can determine $I_4(z)\in S(V_4)^{\Theta_+}$.

Proposition 6.5. *For the field $\mathcal{H}(z)$, there are following OPE relations*

$$\begin{cases} \widehat{e}(z)\mathcal{H}(w) \sim -\frac{6\widehat{e}(w)}{(z-w)^2}, \\ \widehat{f}(z)\mathcal{H}(w) \sim -\frac{6\widehat{f}(w)}{(z-w)^2}, \\ \widehat{h}(z)\mathcal{H}(w) \sim -\frac{6\widehat{h}(w)}{(z-w)^2}, \end{cases}$$

Proof. Using Lemma 6.2 6.3, we shall use relations (6.3)–(6.6) to calculate above OPE relations.

Firstly, we calculate the OPE relations of $\widehat{e}(z)\mathcal{H}(w)$. Since $\mathcal{H}(w)$ has 23 monomials, we need to calculate each monomial. For $\widehat{e}(z) = -\sum_{i=1}^{4} i :\beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z):$, there are

$$\widehat{e}(z)(-\frac{8}{3}:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):) \sim \frac{8}{3}(\frac{-2:\beta^{\varphi_1}(w)\beta^{\varphi_0}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$+\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}+\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}),$$
$$\widehat{e}(z):\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):\sim -(\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}-\frac{4:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w)):\sim 4(\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}),$$
$$\widehat{e}(z)(\frac{2}{3}:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):)\sim -\frac{2}{3}(\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$+\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{3:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-4:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w)):\sim 4(\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}),$$
$$\widehat{e}(z)(\frac{1}{9}:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w)):\sim -\frac{1}{9}(\frac{6:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}),$$

$$\widehat{e}(z)(\frac{8}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):)\sim -\frac{8}{3}(\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$+\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w)):\sim \frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}),$$
$$\widehat{e}(z)(4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):)\sim -4(\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{:\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-\frac{2}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):)\sim \frac{2}{3}(\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$+\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{2:\beta^{\varphi_1}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$

$$\widehat{e}(z)(-2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):)\sim 2(\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}),$$
$$\widehat{e}(z)(\frac{8}{9}:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):)\sim -\frac{8}{9}(\frac{6:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{6:\beta^{\varphi_1}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):)\sim 4(\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}),$$

$$\widehat{e}(z)(\frac{2}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):)\sim -\frac{2}{3}(\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{4:\beta^{\varphi_1}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(4:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):)\sim -4(\frac{:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}),$$
$$\widehat{e}(z)(2:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):)\sim -2(\frac{2:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):)\sim (\frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}),$$

$$\widehat{e}(z)(4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):)\sim -4(\frac{2:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-\frac{2}{3}:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):)\sim \frac{2}{3}(\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-\frac{8}{3}:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim \frac{8}{3}(\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}),$$

$$\widehat{e}(z)(:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim-(\frac{8:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{8:\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(\frac{8}{3}:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim-\frac{8}{3}(\frac{3:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{6:\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):}{(z-w)^2}),$$
$$\widehat{e}(z)(-:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim(\frac{8:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}),$$

finally, there is the OPE relation

$$\begin{aligned}\widehat{e}(z)\mathcal{H}(w)&\sim\frac{6:\beta^{\varphi_0}(w)\gamma^{\xi_1}(w):}{(z-w)^2}+\frac{12:\beta^{\varphi_1}(w)\gamma^{\xi_2}(w):}{(z-w)^2}\\&+\frac{18:\beta^{\varphi_2}(w)\gamma^{\xi_3}(w):}{(z-w)^2}+\frac{24:\beta^{\varphi_3}(w)\gamma^{\xi_4}(w):}{(z-w)^2}\\&=-\frac{6\widehat{e}(w)}{(z-w)^2}.\end{aligned}$$

Analogue to above calculus, we can obtain the following OPE relations.

$$\begin{cases}\widehat{f}(z)\mathcal{H}(w)\sim-\dfrac{6\widehat{f}(w)}{(z-w)^2},\\ \widehat{h}(z)\mathcal{H}(w)\sim-\dfrac{6\widehat{h}(w)}{(z-w)^2}.\end{cases}$$

For $\widehat{f}(z)=-\sum_{i=0}^{3}(4-i):\beta^{\varphi_{i+1}}(z)\gamma^{\xi_i}(z):$, there are

$$\widehat{f}(z)(-\frac{8}{3}:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):)\sim\frac{8}{3}(\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_1}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}),$$
$$\widehat{f}(z):\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):\sim-(\frac{8:\beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_1}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}-\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}$$
$$-\frac{8:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{(z-w)^2}),$$

$$\widehat{f}(z)(-4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w)) :\sim 4(\frac{2:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}),$$
$$\widehat{f}(z)(\frac{2}{3}:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):)\sim -\frac{2}{3}(\frac{4:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$+\frac{3:\beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{4:\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(-4:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w)) :\sim 4(\frac{:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}),$$
$$\widehat{f}(z)(\frac{1}{9}:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w)) :\sim -\frac{1}{9}(\frac{6:\beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}),$$

$$\widehat{f}(z)(\frac{8}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):)\sim -\frac{8}{3}(\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_1}(w):}{z-w}$$
$$-\frac{6:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{6:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(-:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w)) :\sim \frac{8:\beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_1}(w):}{z-w}$$
$$-\frac{6:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}),$$
$$\widehat{f}(z)(4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):)\sim -4(\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(-\frac{2}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):)\sim \frac{2}{3}(\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$+\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{(z-w)^2}),$$

$$\widehat{f}(z)(-2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):) \sim 2(\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}),$$
$$\widehat{f}(z)(\frac{8}{9}:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):) \sim -\frac{8}{9}(\frac{6:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{6:\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$

$$\widehat{f}(z)(-4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):) \sim 4(\frac{4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}),$$
$$\widehat{f}(z)(\frac{2}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):) \sim -\frac{2}{3}(\frac{3:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{2:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{3:\beta^{\varphi_4}(w)\gamma^{\xi_3}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(4:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):) \sim -4(\frac{:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}),$$
$$\widehat{f}(z)(2:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):) \sim -2(\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{2:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(-:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):) \sim (\frac{6:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w):}{z-w}$$
$$-\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}),$$

$$\widehat{f}(z)(4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):)\sim -4(\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$+\frac{:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}-\frac{:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{(z-w)^2}),$$

$$\widehat{f}(z)(-\frac{2}{3}:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):)\sim \frac{2}{3}(\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$+\frac{2:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):}{z-w}-\frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(-\frac{8}{3}:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim \frac{8}{3}(\frac{3:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$+\frac{:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}),$$
$$\widehat{f}(z)(:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim -(\frac{4:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{4:\beta^{\varphi_4}(w)\gamma^{\xi_3}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(\frac{8}{3}:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim -\frac{8}{3}(\frac{3:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$+\frac{:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}-\frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{f}(z)(-:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim (\frac{4:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):}{z-w}$$
$$-\frac{2:\beta^{\varphi_3}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):}{z-w}),$$

then we get

$$\begin{aligned}\widehat{f}(z)\mathcal{H}(w)&\sim \frac{6:\beta^{\varphi_4}(w)\gamma^{\xi_3}(w):}{(z-w)^2}+\frac{12:\beta^{\varphi_3}(w)\gamma^{\xi_2}(w):}{(z-w)^2}\\&+\frac{18:\beta^{\varphi_2}(w)\gamma^{\xi_1}(w):}{(z-w)^2}+\frac{24:\beta^{\varphi_1}(w)\gamma^{\xi_0}(w):}{(z-w)^2}\\&=-\frac{6\widehat{f}(w)}{(z-w)^2}.\end{aligned}$$

For $\widehat{h}(z) = -\sum_{i=0}^{4}(4-2i) : \beta^{\varphi_i}(z)\gamma^{\xi_i}(z) :$, there are

$$\widehat{h}(z)(-\frac{8}{3} : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :) \sim \frac{8}{3}(\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :}{z-w}$$
$$-\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :}{z-w} - \frac{4 : \beta^{\varphi_2}(w)\gamma^{\xi_2}(w) :}{(z-w)^2})$$
$$= -\frac{32}{3}\frac{: \beta^{\varphi_2}(w)\gamma^{\xi_2}(w) :}{(z-w)^2},$$

$$\widehat{h}(z) : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w) :\sim -(\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w) :}{z-w}$$
$$-\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w) :}{z-w}) = 0,$$

$$\widehat{h}(z)(-4 : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w)) :\sim 4(\frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w}$$
$$-\frac{2 : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w} - \frac{4 : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w}$$
$$+\frac{2 : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :}{z-w} - \frac{4 : \beta^{\varphi_3}(w)\gamma^{\xi_3}(w) :}{(z-w)^2} + \frac{2 : \beta^{\varphi_0}(w)\gamma^{\xi_0}(w) :}{(z-w)^2})$$
$$= -\frac{16 : \beta^{\varphi_3}(w)\gamma^{\xi_3}(w) :}{(z-w)^2} + \frac{8 : \beta^{\varphi_0}(w)\gamma^{\xi_0}(w) :}{(z-w)^2},$$

$$\widehat{h}(z)(\frac{2}{3} : \beta^{\varphi_0}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w) :) \sim 0,$$
$$\widehat{h}(z)(-4 : \beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w)) :\sim -\frac{16 : \beta^{\varphi_4}(w)\gamma^{\xi_4}(w) :}{(z-w)^2}$$
$$+\frac{16 : \beta^{\varphi_0}(w)\gamma^{\xi_0}(w) :}{(z-w)^2},$$
$$\widehat{h}(z)(\frac{1}{9} : \beta^{\varphi_0}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w)) :\sim 0,$$
$$\widehat{h}(z)(\frac{8}{3} : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_0}(w)\gamma^{\xi_2}(w) :) \sim 0,$$
$$\widehat{h}(z)(- : \beta^{\varphi_1}(w)\beta^{\varphi_1}(w)\gamma^{\xi_1}(w)\gamma^{\xi_1}(w)) :\sim -\frac{8 : \beta^{\varphi_1}(w)\gamma^{\xi_1}(w) :}{(z-w)^2},$$
$$\widehat{h}(z)(4 : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_3}(w) :) \sim 0,$$
$$\widehat{h}(z)(-\frac{2}{3} : \beta^{\varphi_1}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_2}(w) :) \sim -\frac{4}{3}\frac{: \beta^{\varphi_2}(w)\gamma^{\xi_2}(w) :}{(z-w)^2},$$
$$\widehat{h}(z)(-2 : \beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w) :) \sim -4\frac{: \beta^{\varphi_3}(w)\gamma^{\xi_3}(w) :}{(z-w)^2}$$
$$+4\frac{: \beta^{\varphi_1}(w)\gamma^{\xi_1}(w) :}{(z-w)^2},$$

$$\widehat{h}(z)(\frac{8}{9}:\beta^{\varphi_1}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):)\sim 0,$$
$$\widehat{h}(z)(-4:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):)\sim -8\frac{:\beta^{\varphi_4}(w)\gamma^{\xi_4}(w):}{(z-w)^2}$$
$$+16\frac{:\beta^{\varphi_1}(w)\gamma^{\xi_1}(w):}{(z-w)^2},$$
$$\widehat{h}(z)(\frac{2}{3}:\beta^{\varphi_1}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):)\sim 0,$$
$$\widehat{h}(z)(4:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_0}(w)\gamma^{\xi_4}(w):)\sim 0,$$

$$\widehat{h}(z)(2:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_1}(w)\gamma^{\xi_3}(w):)\sim 0,$$
$$\widehat{h}(z)(-:\beta^{\varphi_2}(w)\beta^{\varphi_2}(w)\gamma^{\xi_2}(w)\gamma^{\xi_2}(w):)\sim 0,$$
$$\widehat{h}(z)(4:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_1}(w)\gamma^{\xi_4}(w):)\sim 0,$$
$$\widehat{h}(z)(-\frac{2}{3}:\beta^{\varphi_2}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_3}(w):)\sim \frac{4}{3}\frac{:\beta^{\varphi_2}(w)\gamma^{\xi_2}(w):}{(z-w)^2}),$$
$$\widehat{h}(z)(-\frac{8}{3}:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim \frac{32}{3}\frac{:\beta^{\varphi_2}(w)\gamma^{\xi_2}(w):}{(z-w)^2},$$
$$\widehat{h}(z)(:\beta^{\varphi_2}(w)\beta^{\varphi_4}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim 0,$$
$$\widehat{h}(z)(\frac{8}{3}:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_2}(w)\gamma^{\xi_4}(w):)\sim 0,$$
$$\widehat{h}(z)(-:\beta^{\varphi_3}(w)\beta^{\varphi_3}(w)\gamma^{\xi_3}(w)\gamma^{\xi_3}(w):)\sim 8\frac{:\beta^{\varphi_3}(w)\gamma^{\xi_3}(w):}{(z-w)^2},$$

finally, there is the OPE relation

$$\widehat{h}(z)\mathcal{H}(w)\sim \frac{24:\beta^{\varphi_0}(w)\gamma^{\xi_0}(w):}{(z-w)^2}+\frac{12:\beta^{\varphi_1}(w)\gamma^{\xi_1}(w):}{(z-w)^2}$$
$$-\frac{12:\beta^{\varphi_3}(w)\gamma^{\xi_3}(w):}{(z-w)^2}-\frac{24:\beta^{\varphi_4}(w)\gamma^{\xi_4}(w):}{(z-w)^2}=-\frac{6\widehat{h}(w)}{(z-w)^2}.$$

Proposition 6.6. *Let* $\triangle(z)=\sum_{j=0}^{4}\big(:\partial\beta^{\varphi_j}(z)\gamma^{\xi_j}(z):-:\beta^{\varphi_j}(z)\partial\gamma^{\xi_j}(z):\big)$, *then there are the following OPE relations*

$$\begin{cases}\widehat{e}(z)\triangle(w)\sim -\dfrac{2\widehat{e}(w)}{(z-w)^2},\\ \widehat{f}(z)\triangle(w)\sim -\dfrac{2\widehat{f}(w)}{(z-w)^2},\\ \widehat{h}(z)\triangle(w)\sim -\dfrac{2\widehat{h}(w)}{(z-w)^2},\end{cases}$$

Proof. According to Lemma 6.2, 6.3, we shall use relations (6.3)–(6.8) to calculate above OPE relations.

At first, using Lemma 6.3, we have

$$\begin{aligned}
\widehat{e}(z)\triangle(w) &= -\sum_{i=1}^{4} i : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z) : \triangle(w) \\
&= -\sum_{i=1}^{4}\sum_{j=0}^{4} i(: \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)(\partial\beta^{\varphi_j}(w)\gamma^{\xi_j}(w) - \beta^{\varphi_j}(w)\partial\gamma^{\xi_j}(w)) : \\
&+ \frac{\delta_{i-1,j} : \partial\beta^{\varphi_j}(w)\gamma^{\xi_i}(z) :}{z-w} - \frac{\delta_{i,j} : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_j}(w) :}{(z-w)^2} \\
&- \frac{\delta_{i-1,j} : \beta^{\varphi_j}(w)\gamma^{\xi_i}(z) :}{(z-w)^2} + \frac{\delta_{i,j} : \beta^{\varphi_{i-1}}(z)\partial\gamma^{\xi_j}(w) :}{z-w}) \\
&= -\sum_{i=1}^{4} i(\sum_{j=0}^{4} : \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(z)(\partial\beta^{\varphi_j}(w)\gamma^{\xi_j}(w) - \beta^{\varphi_j}(w)\partial\gamma^{\xi_j}(w)) \\
&+ : \frac{: \partial\beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(z) :}{z-w} - \frac{: \beta^{\varphi_{i-1}}(z)\gamma^{\xi_i}(w) :}{(z-w)^2} \\
&- \frac{: \beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(z) :}{(z-w)^2} + \frac{: \beta^{\varphi_{i-1}}(z)\partial\gamma^{\xi_i}(w) :}{z-w}).
\end{aligned}$$

Using Lemma 6.2, there is

$$\begin{aligned}
&\widehat{e}(z)\triangle(w) \sim \\
&-\sum_{i=1}^{4} i(\frac{: \partial\beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(w) :}{z-w} - \frac{: \beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(w) : +(z-w) : \partial\beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(w) :}{(z-w)^2} \\
&- \frac{: \beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(w) : +(z-w) : \beta^{\varphi_{i-1}}(w)\partial\gamma^{\xi_i}(w) :}{(z-w)^2} + \frac{: \beta^{\varphi_{i-1}}(w)\partial\gamma^{\xi_i}(w) :}{z-w}) \\
&= -\sum_{i=1}^{4} i(\frac{-2 : \beta^{\varphi_{i-1}}(w)\gamma^{\xi_i}(w) :}{(z-w)^2}) \\
&= \frac{-2\widehat{e}(w)}{(z-w)^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\widehat{h}(z)\triangle(w) &= -\sum_{i=0}^{4}(4-2i) : \beta^{\varphi_i}(z)\gamma^{\xi_i}(z) : \triangle(w) \\
&= -\sum_{i=0}^{4}(4-2i)(\sum_{j=0}^{4} : \beta^{\varphi_i}(z)\gamma^{\xi_i}(z)(\partial\beta^{\varphi_j}(w)\gamma^{\xi_j}(w) - \beta^{\varphi_j}(w)\partial\gamma^{\xi_j}(w))
\end{aligned}$$

$$+\frac{:\partial\beta^{\varphi_i}(w)\gamma^{\xi_i}(z):}{(z-w)}-\frac{:\beta^{\varphi_i}(z)\gamma^{\xi_i}(w):}{(z-w)^2}-\frac{1}{(z-w)^3}$$
$$-\frac{:\beta^{\varphi_i}(w)\gamma^{\xi_i}(z):}{(z-w)^2}+\frac{:\beta^{\varphi_i}(z)\partial\gamma^{\xi_i}(w):}{(z-w)}+\frac{1}{(z-w)^3})$$
$$=-\sum_{i=0}^{4}(4-2i)(\sum_{j=0}^{4}:\beta^{\varphi_i}(z)\gamma^{\xi_i}(z)(\partial\beta^{\varphi_j}(w)\gamma^{\xi_j}(w)-\beta^{\varphi_j}(w)\partial\gamma^{\xi_j}(w))$$
$$+\frac{:\partial\beta^{\varphi_i}(w)\gamma^{\xi_i}(z):}{(z-w)}-\frac{:\beta^{\varphi_i}(z)\gamma^{\xi_i}(w):}{(z-w)^2}$$
$$-\frac{:\beta^{\varphi_i}(w)\gamma^{\xi_i}(z):}{(z-w)^2}+\frac{:\beta^{\varphi_i}(z)\partial\gamma^{\xi_i}(w):}{(z-w)}),$$

then there is

$$\widehat{h}(z)\triangle(w)=-\sum_{i=0}^{4}(4-2i):\beta^{\varphi_i}(z)\gamma^{\xi_i}(z):\triangle(w)$$
$$\sim-\sum_{i=0}^{4}(4-2i)(\frac{:\partial\beta^{\varphi_i}(w)\gamma^{\xi_i}(w):}{(z-w)}-\frac{:\beta^{\varphi_i}(w)\gamma^{\xi_i}(w):+(z-w):\partial\beta^{\varphi_i}(w)\gamma^{\xi_i}(w):}{(z-w)^2}$$
$$-\frac{:\beta^{\varphi_i}(w)\gamma^{\xi_i}(w):+:\beta^{\varphi_i}(w)\partial\gamma^{\xi_i}(w):}{(z-w)^2}+\frac{:\beta^{\varphi_i}(w)\partial\gamma^{\xi_i}(w):}{(z-w)})$$
$$=\sum_{i=0}^{4}2(4-2i)\frac{:\beta^{\varphi_i}(w)\gamma^{\xi_i}(w):}{(z-w)^2}$$
$$=\frac{-2\widehat{h}(w)}{(z-w)^2}.$$

Meanwhile, we have

$$\widehat{f}(z)\triangle(w)=-\sum_{i=0}^{3}(4-i):\beta^{\varphi_{i+1}}(z)\gamma^{\xi_i}(z):\triangle(w)$$
$$=-\sum_{i=0}^{3}(4-i)(\sum_{j=0}^{4}:\beta^{\varphi_{i+1}}(z)\gamma^{\xi_i}(z)(\partial\beta^{\varphi_j}(w)\gamma^{\xi_j}(w)-\beta^{\varphi_j}(w)\partial\gamma^{\xi_j}(w))$$
$$+\frac{:\partial\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(z):}{z-w}-\frac{:\beta^{\varphi_{i+1}}(z)\gamma^{\xi_i}(w):}{(z-w)^2}$$
$$-\frac{:\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(z):}{(z-w)^2}+\frac{:\beta^{\varphi_{i+1}}(z)\partial\gamma^{\xi_i}(w):}{z-w}),$$

then there is

$$
\begin{aligned}
&\widehat{f}(z)\triangle(w)\\
&\sim -\sum_{i=0}^{3}(4-i)\Big(\frac{:\partial\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(w):}{z-w}-\frac{:\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(w):+(z-w):\partial\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(w):}{(z-w)^2}\\
&-\frac{:\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(w):+(z-w):\beta^{\varphi_{i+1}}(w)\partial\gamma^{\xi_i}(w):}{(z-w)^2}\Big)\\
&=-\sum_{i=0}^{3}(4-i)\Big(\frac{-2:\beta^{\varphi_{i+1}}(w)\gamma^{\xi_i}(w):}{(z-w)^2}\Big)\\
&=\frac{-2\widehat{f}(w)}{(z-w)^2}.
\end{aligned}
$$

Corollary 6.7. *The field $\mathcal{H}(z)-3\triangle(z)\in S(V_4)^{\Theta_+}$.*

Proof. From Proposition 6.5, 6.6, we have

$$
\begin{cases}
\widehat{e}(z)(\mathcal{H}(w)-3\triangle(w))\sim 0,\\
\widehat{f}(z)(\mathcal{H}(w)-3\triangle(w))\sim 0,\\
\widehat{h}(z)(\mathcal{H}(w)-3\triangle(w))\sim 0,
\end{cases}
$$

so $\mathcal{H}(w)-3\triangle(w)\in S(V_4)^{\Theta_+}$.

Theorem 6.8. *The commutant $S(V_4)^{\Theta_+}$ is strongly generated by the finite set*

$$\{v^e(z),v^h(z),v^f(z),I_1(z),I_2(z),I_3(z),I_4(z),\mathcal{H}(z)-3\triangle(z)\}.$$

Proof. According to Propostion 6.1, 6.4, and 6.7, we know that the finite generator set $\{v^e,v^h,v^f,I_1,I_2,I_3,I_4,\mathcal{H}\}$ of $gr(S(V_4))^{\Theta_+}$ is also the finite generator set of $gr(S(V_4)^{\Theta_+})$, i.e. $\Gamma: gr(S(V_4)^{\Theta_+})\hookrightarrow gr(S(V_4))^{\Theta_+}$ is surjective. By the Reconstruction Property (cf. Lemma 4.1), we get that $S(V_4)^{\Theta_+}$is strongly generated by the following finite set $\{v^e(z),v^h(z),v^f(z),I_1(z),I_2(z),I_3(z),I_4(z),\mathcal{H}(z)-3\triangle(z)\}$.

7 The OPE Relations among the Generators of $S(V_4)^{\Theta_+}$

In last section, we shall give the OPE relations among the generators of $S(V_4)^{\Theta_+}$.

About the calculations of these OPE relations, we can calculate by Lemma 6.2, 6.3. Our approach is as follows, at first, according to the formula (6.2) in Lemma 6.3, we write the expansions between each a pair of generators, then we can calculate expansions with singular parts using the commutations relations(6.3)–(6.10) and give reduced expansions. Next, we apply the Taylor's expansions (6.1) into above reduced expansions, we get the OPE relations only consisting of singular parts among these generators by complex calculations. Since it needs almost two hundred papers to give the details of our calculations, we only write down the formulas of OPE relations here. If any one needs the details, we are glad to show you.

In order to express the following OPE relations, we give some symbol conventions. In our case, if there are two fields to be forms as $a(w) = \sum_{i=1}^{N} a_i : \alpha_{i_1}(w)\cdots\alpha_{i_r}(w) :, b(w) = \sum_{j=1}^{M} b_j : \beta_{j_1}(w)\cdots\beta_{j_s}(w) :$, we denote by $a(w) \centerdot b(w) := \sum_{i=1}^{N}\sum_{j=1}^{M} a_i b_j : \alpha_{i_1}(w)\cdots\alpha_{i_r}(w)\beta_{j_1}(w)\cdots\beta_{j_s}(w) :$, where $M, N, r, s \in \mathbb{N}, a_i, b_j \in \mathbb{C}$.

Proposition 7.1. *About the generator $v^e(z)$, there are the OPE relations with other generators*

$$v^e(z)v^e(w) \sim 0, \quad v^e(z)I_1(w) \sim 0, \tag{7.1}$$

$$v^e(z)v^f(w) \sim \frac{v^h(w)}{z-w} - \frac{5}{2(z-w)^2}, \tag{7.2}$$

$$v^e(z)v^h(w) \sim -\frac{2v^e(w)}{z-w}, \tag{7.3}$$

$$v^e(z)I_2(w) \sim -\frac{I_3(w)}{z-w}, \tag{7.4}$$

$$v^e(z)I_3(w) \sim -\frac{I_4(w)}{3(z-w)}, \tag{7.5}$$

$$v^e(z)I_4(w) \sim -\frac{3}{8}\frac{I_1(w)}{z-w}, \tag{7.6}$$

$$v^e(z)\mathcal{H}(w) \sim -\frac{10}{3}\frac{v^e(w)}{(z-w)^2} + \frac{4v^h(w) \centerdot v^e(w)}{z-w}, \tag{7.7}$$

$$v^e(z)\triangle(w) \sim -\frac{2v^e(w)}{(z-w)^2}, \tag{7.8}$$

$$v^e(\mathcal{H}(w) - 3\triangle(w)) \sim \frac{4v^h(w) \centerdot v^h(w)}{z-w} + \frac{8}{3}\frac{v^e(w)}{(z-w)^2}. \tag{7.9}$$

Proposition 7.2. *About the generator $v^f(z)$, there are the OPE relations with other generators as follows*

$$v^f(z)v^f(w) \sim 0, \quad v^f(z)I_2(w) \sim 0, \tag{7.10}$$

$$v^f(z)v^h(w) \sim \frac{2v^f(w)}{z-w}, \tag{7.11}$$

$$v^f(z)I_1(w) \sim -\frac{8I_4(w)}{z-w}, \tag{7.12}$$

$$v^f(z)I_3(w) \sim -\frac{3I_2(w)}{z-w}, \tag{7.13}$$

$$v^f(z)I_4(w) \sim -\frac{12I_3(w)}{z-w}, \tag{7.14}$$

$$v^f(z)\mathcal{H}(w) \sim -\frac{10}{3}\frac{v^f(w)}{(z-w)^2} - \frac{4v^f(w)\,\mathbf{.}\,v^h(w)}{z-w}, \tag{7.15}$$

$$v^f(z)\triangle(w) \sim -\frac{2v^f(w)}{(z-w)^2}, \tag{7.16}$$

$$v^f(z)(\mathcal{H}(w) - 3\triangle(w)) \sim -\frac{4v^f(w)\,\mathbf{.}\,v^h(w)}{z-w} + \frac{8}{3}\frac{v^f(w)}{(z-w)^2}. \tag{7.17}$$

Proposition 7.3. *There are the following OPE relations between the generator $v^h(z)$ and other generators*

$$v^h(z)v^h(w) \sim -\frac{5}{(z-w)^2}, \tag{7.18}$$

$$v^h(z)I_1(w) \sim \frac{3I_1(w)}{z-w}, \tag{7.19}$$

$$v^h(z)I_2(w) \sim -\frac{3I_2(w)}{z-w}, \tag{7.20}$$

$$v^h(z)I_3(w) \sim -\frac{I_3(w)}{z-w}, \tag{7.21}$$

$$v^h(z)I_4(w) \sim \frac{I_4(w)}{z-w}, \tag{7.22}$$

$$v^h(z)\mathcal{H}(w) \sim \frac{32}{3}\frac{v^h(w)}{(z-w)^2}, \tag{7.23}$$

$$v^h(z)\triangle(w) \sim -\frac{2v^h(w)}{(z-w)^2}, \tag{7.24}$$

$$v^h(z)(\mathcal{H}(w) - 3\triangle(w)) \sim \frac{50}{3}\frac{v^h(z)}{(z-w)^2}. \tag{7.25}$$

Proposition 7.4. *There are the following OPE relations between the generator $I_1(z)$ and other generators*

$$I_1(z)I_1(w) \sim 0, \tag{7.26}$$

$$I_1(z)I_2(w) \sim \frac{3\mathcal{H}(w) - 4v^e(w) \centerdot v^f(w) + 7(\partial v^h(w) - \triangle(w))}{z-w} + \frac{14v^h(w)}{(z-w)^2} - \frac{70}{3(z-w)^3}, \tag{7.27}$$

$$I_1(z)I_3(w) \sim \frac{14\partial v^e(w) - 8v^e(w) \centerdot v^h(w)}{z-w} + \frac{28v^e(w)}{(z-w)^2}, \tag{7.28}$$

$$I_1(z)I_4(w) \sim -\frac{48v^e(w) \centerdot v^e(w)}{z-w}, \tag{7.29}$$

$$I_1(z)\mathcal{H}(w) \sim \frac{8v^h(w) \centerdot I_1(w) - \frac{32}{3}v^e(w) \centerdot I_4(w) - \frac{14}{3}\partial I_1(w)}{z-w} - \frac{14I_1(w)}{(z-w)^2}, \tag{7.30}$$

$$I_1(z)\triangle(w) \sim -\frac{\partial I_1(w)}{z-w} - \frac{3I_1(w)}{(z-w)^2}, \tag{7.31}$$

$$I_1(z)(\mathcal{H}(w) - 3\triangle(w)) \sim \frac{8v^h(w) \centerdot I_1(w) - \frac{32}{3}v^e(w) \centerdot I_4(w) - \frac{5}{3}\partial I_1(w)}{z-w} - \frac{5I_1(w)}{(z-w)^2}. \tag{7.32}$$

Proposition 7.5. *About the generator $I_2(z)$, there are the OPE relations with other generators as follows*

$$I_2(z)I_2(w) \sim 0, \tag{7.33}$$

$$I_2(z)I_3(w) \sim \frac{1}{6}\frac{v^f(w) \centerdot v^f(w)}{z-w}, \tag{7.34}$$

$$I_2(z)I_4(w) \sim -\frac{v^f(w) \centerdot v^h(w) + \frac{7}{4}\partial v^f(w)}{z-w} - \frac{7}{2}\frac{v^f(w)}{(z-w)^2}, \tag{7.35}$$

$$I_2(z)\mathcal{H} \sim -\frac{8I_2(w) \centerdot v^h(w) + \frac{4}{3}v^f(w) \centerdot I_3(w) + \frac{14}{3}\partial I_2(w)}{z-w} - \frac{14I_2(w)}{(z-w)^2}, \tag{7.36}$$

$$I_2(z)\triangle(w) \sim -\frac{\partial I_2(w)}{z-w} - \frac{3I_2(w)}{(z-w)^2}, \tag{7.37}$$

$$I_2(z)(\mathcal{H}(w)-3\triangle(w)) \sim -\frac{8I_2(w)\cdot v^h(w)+\frac{4}{3}v^f(w)\cdot I_3(w)+\frac{5}{3}\partial I_2(w)}{z-w} -\frac{5I_2(w)}{(z-w)^2}. \tag{7.38}$$

Proposition 7.6. *About the generator $I_3(z)$, there are the OPE relations with other generators as follows*

$$I_3(z)I_3(w) \sim \frac{7}{12}\frac{\partial v^f(w)}{z-w}+\frac{7}{6}\frac{v^f(w)}{(z-w)^2}, \tag{7.39}$$

$$I_3(z)I_4(w) \sim \frac{\frac{9}{8}\mathcal{H}(w)+v^h(w)\cdot v^h(w)-\frac{7}{2}v^e(w)\cdot v^f(w)+\frac{7}{4}\partial v^h(w)}{z-w} -\frac{21}{8}\frac{\partial v^h(w)+\triangle(w)}{z-w}-\frac{7}{4}\frac{v^h(w)}{(z-w)^2}-\frac{35}{4(z-w)^3}, \tag{7.40}$$

$$I_3(z)\mathcal{H}(w) \sim -\frac{\frac{8}{3}v^h(w)\cdot I_3(w)+\frac{4}{9}v^f(w)\cdot I_4(w)+4I_2(w)\cdot v^e(w)}{z-w} -\frac{\frac{2}{3}\partial I_3(w)+4\partial_3 I_3(w)}{z-w}-\frac{6I_3(w)}{(z-w)^2}, \tag{7.41}$$

$$I_3(z)\triangle(w) \sim -\frac{\partial I_3(w)}{z-w}-\frac{3I_3(w)}{(z-w)^2}, \tag{7.42}$$

$$I_3(z)(\mathcal{H}(w)-3\triangle(w)) \sim -\frac{\frac{8}{3}v^h(w)\cdot I_3(w)+\frac{4}{9}v^f(w)\cdot I_4(w)+4I_2(w)\cdot v^e(w)}{z-w} +\frac{\frac{7}{3}\partial I_3(w)-4\partial_3 I_3(w)}{z-w}+\frac{3I_3(w)}{(z-w)^2}, \tag{7.43}$$

where $\partial_3 I_3(w)$ means that for each monomials of $I_3(w)$, we solve the formal derivative for the field corresponding to the third position, example for , if $I(w) =: \alpha_1(w)\cdots\alpha_N(w):$, we denote $:\alpha_1(w)\cdots\partial\alpha_k(w)\cdots\alpha_N(w):$ by $\partial_k I(w)$.

Proposition 7.7. *There are the following OPE relations between the generator $I_4(z)$ and other generators*

$$I_4(z)I_4(w) \sim -\frac{21\partial v^e(w)}{z-w}-\frac{42v^e(w)}{(z-w)^2}, \tag{7.44}$$

$$I_4(z)\mathcal{H}(w) \sim \frac{\frac{8}{3}v^h(w)\,.\,I_4(w) - \frac{1}{2}v^f(w)\,.\,I_1(w) - 16I_3(w)\,.\,v^e(w)}{z-w} - \frac{\frac{2}{3}\partial I_4(w) + 4\partial_1 I_4(w)}{z-w} - \frac{6I_4(w)}{(z-w)^2}, \tag{7.45}$$

$$I_4(z)\triangle(w) \sim -\frac{\partial I_4(w)}{z-w} - \frac{3I_4(w)}{(z-w)^2}, \tag{7.46}$$

$$I_4(z)(\mathcal{H}(w) - 3\triangle(w)) \sim \frac{\frac{8}{3}v^h(w)\,.\,I_4(w) - \frac{1}{2}v^f(w)\,.\,I_1(w) - 16I_3(w)\,.\,v^e(w)}{z-w} + \frac{\frac{7}{3}\partial I_4(w) - 4\partial_1 I_4(w)}{z-w} + \frac{3I_4(w)}{(z-w)^2}, \tag{7.47}$$

where $\partial_1 I_4(w)$ is the formal derivative for the field corresponding to the first position.

Proposition 7.8. *About the generator $\mathcal{H}(w) - 3\triangle(w)$, there are the following OPE relations*

$$\mathcal{H}(z)\mathcal{H}(w) \sim \frac{-6\partial\mathcal{H}(w) - \frac{176}{9}\partial(v^h(w)\,.\,v^h(w)) - \frac{80}{9}\partial(v^e(w)\,.\,v^f(w)) + 48\partial\triangle(w)}{z-w} - \frac{12\mathcal{H}(w) + \frac{352}{9}v^h(w)\,.\,v^h(w) + \frac{160}{9}v^e(w)\,.\,v^f(w) - 48\triangle(w)}{(z-w)^2} + \frac{120}{(z-w)^4}, \tag{7.48}$$

$$\mathcal{H}(z)\triangle(w) \sim -2\frac{\partial\mathcal{H}(w)}{z-w} - \frac{4\mathcal{H}(w)}{(z-w)^2}, \tag{7.49}$$

$$\triangle(z)\mathcal{H}(w) \sim -\frac{2\partial\mathcal{H}(w)}{z-w} - \frac{4\mathcal{H}(w)}{(z-w)^2}, \tag{7.50}$$

$$\triangle(z)\triangle(w) \sim -\frac{3\partial\triangle(w)}{z-w} - \frac{4\triangle(w)}{(z-w)^2} - \frac{10}{(z-w)^4}, \tag{7.51}$$

$$(\mathcal{H}(z) - 3\triangle(z))(\mathcal{H}(w) - 3\triangle(w)) \sim \frac{6\partial\mathcal{H}(w) - \frac{176}{9}\partial(v^h(w)\,.\,v^h(w)) - \frac{80}{9}\partial(v^e(w)\,.\,v^f(w)) + 21\partial\triangle(w)}{z-w} + \frac{12\mathcal{H}(w) - \frac{352}{9}v^h(w)\,.\,v^h(w) - \frac{160}{9}v^e(w)\,.\,v^f(w) + 12\triangle(w)}{(z-w)^2} + \frac{30}{(z-w)^4}. \tag{7.52}$$

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